A CLIFFORDIAN MONOPOLE EQUATION AND SMOOTH INVARIANTS OF A SPIN MANIFOLD

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Abstract. We construct a non-linear elliptic partial differential equation for a given smooth closed spin $4k$-manifold $(k \geq 1)$ and a Hermitian line bundle over the manifold. The equation is an extended version of an analogue of the Seiberg-Witten equation for a smooth closed Spin^c 4-manifold. We define smooth invariants of the $4k$-manifold by using the moduli space of all solutions of the equation.

1. Introduction

Let X be a smooth oriented closed manifold with a Riemannian metric g. Let E and F be smooth vector bundles of finite rank over X with Hermitian metrics h_E and h_F respectively. We set $V = \Gamma(E)$ and $W = \Gamma(F)$. Let G be a compact Lie group. We assume that V and W be right G-modules on which G acts orthogonally. We consider a G-equivariant non-linear elliptic operator of the form,

$$D + Q\colon V \longrightarrow W,$$

where D is a 1st order linear elliptic operator and Q is a quadratic map, i.e., there exists a bi-linear map $\hat{Q}\colon V \times V \longrightarrow W$ such that $Q(v) = \hat{Q}(v, v)$ for all $v \in V$. The space $\mathcal{S}$ of all solutions of the equation $(D + Q)(v) = 0$ for $v \in V$ is $(D + Q)^{-1}(0)$. The moduli space $\mathcal{M}$ of $\mathcal{S}$ is defined as $(D+Q)^{-1}(0)/G$. We would like to construct global differential topological invariants of X by using the data of $\mathcal{S}$ and $\mathcal{M}$, which depend only on the isomorphism classes of E and F, and do not depend on the Riemannian metric g and the Hermitian metrics h_E, h_F. Let us consider the case where both $\mathcal{S}$ and $\mathcal{M}$ are compact. Then we have the following family of linear elliptic operators parameterized by $\mathcal{S}$,

$$\mathcal{D} = \left\{ D_v = D + \hat{Q}(v, \cdot)\colon V \longrightarrow W \mid v \in \mathcal{S} \right\}.$$

The index bundle of $\mathcal{D}$ is a virtual G-bundle over $\mathcal{S}$, i.e., $\operatorname{index} \mathcal{D} \in K_G(\mathcal{S})$. We set $\operatorname{index} \hat{\mathcal{D}} = \operatorname{index} \mathcal{D}/G$. Then $\operatorname{index} \hat{\mathcal{D}}$ is a virtual bundle over $\mathcal{M}$, i.e., $\operatorname{index} \hat{\mathcal{D}} \in K(\mathcal{M})$. We expect that the number

$$q_s(X, E, F) = \langle ch(\operatorname{index} \hat{\mathcal{D}}), [\mathcal{M}] \rangle$$

is such an invariant, where $ch\colon K(\mathcal{M}) \longrightarrow \mathrm{H}^*(\mathcal{M}; \mathbb{Q})$ is the Chern character homomorphism and the suffix s is an auxiliary global geometrical structure of X, e.g. spin structure,

Date: 2002/6/1,The latest version 2005/6/11.

which is used in order to construct an elliptic operator $D+Q$. Furthermore, if the compact Lie group G acts on $\mathcal{S}$ freely, then $\mathcal{S}$ has a principal G-bundle structure over $\mathcal{M}$. Then the number

$$q''_s(X,E,F) = \langle ch(\mathcal{S}), [\mathcal{M}]\rangle$$

be expected to be such an invariant of X. On the other hand, by using the method of "finite dimensional approximation of the map $D+Q$" developed by Furuta [14],[15], we may construct such an invariant as a stable homotopy class of maps,

$$\phi_s(X,E,F) \in [S(V), S(W)]^G,$$

where

$$\phi_s(X,E,F) = \varinjlim_{\lambda}(D_\lambda + Q_\lambda), \quad [S(V),S(W)]^G = \varinjlim_{\lambda}\,[S(V_\lambda), S(W_\lambda)]^G$$

and

$$D_\lambda + Q_\lambda \colon V_\lambda \longrightarrow W_\lambda$$

is a finite dimensional approximation of $(D+Q)\colon V \longrightarrow W$ and $S(V_\lambda)$ and $S(W_\lambda)$ are spheres in V_λ and W_λ respectively.

Our problem is how to construct an instance of the above model $D+Q\colon V \longrightarrow W$ such that $\mathcal{S}$ and $\mathcal{M}$ are compact for a manifold X in some suitable category with an auxiliary global geometrical structure. We already have an example, the Seiberg-Witten equation reformulated by Furuta [14]. In Furuta's theory, X is a closed spin 4-manifold and $V = (\Omega^1 \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}) \oplus \varGamma(S^+ \otimes_{\mathbb{C}} L)$ and $W = ((\Omega^0 \oplus \Omega_+) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}) \oplus \varGamma(S^- \otimes_{\mathbb{C}} L)$, where Ω^* is the space of differential forms on X and Ω_+ is the space of self-dual 2-forms and $S = S^+ \oplus S^-$ is the complex spinor bundle for the spin structure and L is a trivial Hermitian line bundle over X. The linear elliptic operator D is given by

$$D = \begin{pmatrix} d^+ + d^* & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix},$$

where $d^+ + d^*$ is equivalent to the AHS complex

$$0 \longrightarrow \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d^+\ } \Omega^2_+ \longrightarrow 0$$

and $\partial\!\!\!/$ is the half of a twisted Dirac operator. The quadratic map Q is given by

$$Q\begin{pmatrix} a \\ \phi \end{pmatrix} = \begin{pmatrix} \phi\sqrt{-1}\bar{\phi} \\ \sqrt{-1}c(a)\phi \end{pmatrix},$$

where $c\colon \Omega^* \longrightarrow C(X)$ is the quantization isomorphism to the Clifford bundle and $c(a)\phi$ means the Clifford multiplication and $\phi\sqrt{-1}\bar{\phi}$ means the multiplication in the quaternions $\mathbb{H}$ which is considered as the Cliffordian multiplication. In this theory, the auxiliary geometrical structure s is a spin structure on X and the compact Lie group is $\mathrm{Pin}(2) = \langle S^1, j\rangle \subset \mathbb{H}^\times$. The Seiberg-Witten equation is given by the Pin(2)-equivariant non-linear elliptic operator $(D+Q)\colon V \longrightarrow W$. The moduli space $\mathcal{M} = (D+Q)^{-1}(0)/S^1$ is a smooth oriented closed manifold. The Seiberg-Witten invariant is given by

$$q''_s(X,L) = \int_{\mathcal{M}} c_1(\mathcal{S})^d,$$

where $d = \frac{1}{2}\dim\mathcal{M}$. Let us denote by $Spin(X)$ the space of spin structures on X. The map

$$\begin{array}{ccc} Spin(X,L) & \xrightarrow{q''} & \mathbb{Z} \\ s & \longmapsto & q''_s(X,L) \end{array}$$

is an invariant of orientation preserving diffeomorphisms of X. The stable homotopy version of the Seiberg-Witten invariant defined by Furuta is given by

$$\phi_s(X) = \lim_{\overrightarrow{\lambda}}\,[D_\lambda + Q_\lambda] \in \lim_{\overrightarrow{x,y}} \left[S(\mathbb{H}^{x+k} \oplus \tilde{\mathbb{R}}^y), S(\mathbb{H}^x \oplus \tilde{\mathbb{R}}^{l+y})\right]^{\mathrm{Pin}(2)},$$

where $kE_8 \oplus lH$ is the intersection form of X and $\tilde{\mathbb{R}}$ is the non-trivial real 1-dimensional $\mathrm{Pin}(2)$-module.

We construct our theory as an analogue of Furuta's theory. Let X be a connected closed oriented spin manifold of dimension $n = 4k \quad (k \geq 1)$ with a Riemannian metric g. Let L be a Hermitian line bundle over X with a Hermitian metric h and let P_L be a principal $\mathrm{U}(1) = S^1$-bundle over X such that $P_L \times_{\mathrm{U}(1)} \mathbb{C} \cong L$. The real Clifford bundle $C(X) = C^+(X) \oplus C^-(X)$ is a $\mathbb{Z}_2$-grading (super) vector bundle over X and the complex spinor bundle $S = S^+ \oplus S^-$ for a given spin structure of X is a $\mathbb{Z}_2$-grading $C(X)$-module. We denote by γ the chirality operator. We set $C_\pm(X) = \frac{1\pm\gamma}{2}C(X)$. The spaces $\frac{1\pm\gamma}{2}C(X)$ are the ±1-eigen spaces of $C(X)$ for the chirality operator γ respectively. We have the decomposition of $C(X)$:

$$C(X) = C^+_+(X) \oplus C^+_-(X) \oplus C^-_+(X) \oplus C^-_-(X).$$

We can consider the quarter of a twisted Dirac operator

$$\partial\!\!\!/_C \colon \Gamma(C^-_-(X) \otimes_{\mathbb{R}} AdP_L) \longrightarrow \Gamma(C^+_+(X) \otimes_{\mathbb{R}} AdP_L),$$

which is equivalent to the extended AHS-complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2k-1} \xrightarrow{d^+} \Omega^{2k}_+ \longrightarrow 0,$$

where Ω^{2k}_+ is the space of self-dual $2k$-forms. We construct our equation using the operator $\partial\!\!\!/_C$ instead of the operator $d^* + d^+$ in the Seiberg-Witten equation. We denote by s the spin structure of X. We define the Cliffordian monopole equation $D + Q \colon V \longrightarrow W$ for X, L, and s as follows: Let $V = \Gamma(S^+ \otimes_{\mathbb{C}} L) \oplus \Gamma(C^-_- \otimes_{\mathbb{R}} AdP_L)$ and $W = \Gamma(S^- \otimes_{\mathbb{C}} L) \oplus \Gamma(C^+_+ \otimes_{\mathbb{R}} AdP_L)$, where P_L is the principal $U(1)$-bundle associated with L. The operator $D \colon V \longrightarrow W$ in our theory is defined by

$$D = \begin{pmatrix} \partial\!\!\!/_S & 0 \\ 0 & \partial\!\!\!/_C \end{pmatrix},$$

where $\partial\!\!\!/_S\colon \Gamma(S^+\otimes_{\mathbb{C}}L) \longrightarrow \Gamma(S^-\otimes_{\mathbb{C}}L)$ is the half of twisted Dirac operator. The quadratic map Q is defined by

$$Q\begin{pmatrix}\phi\\ \xi\end{pmatrix} = \begin{pmatrix}-\xi\phi\\ (\phi\otimes\phi^*)_0 + \sqrt{-1}\xi_{\#}\xi\end{pmatrix},$$

where $\phi \in \Gamma(S^+\otimes_{\mathbb{C}}L)$, $\xi \in \Gamma(C_-^-\otimes_{\mathbb{R}}AdP_L)$, $\xi_{\#} \in \Gamma(C_+^-\otimes_{\mathbb{R}}AdP_L)$ and $(\phi\otimes\phi^*)_0 \in \Gamma(C_+^+\otimes_{\mathbb{R}}AdP_L)$ means the purely imaginary part of $\phi\otimes\phi^* \in \Gamma(C_+^+\otimes_{\mathbb{R}}\mathbb{C})$. The #-operator is defined in Section 2. Here we call our equation "$(D+Q)(\phi,\xi) = 0$" the Cliffordian monopole equation for the pair (X, L). The operator $D + Q$ is S^1-equivariant. The perturbed Cliffordian monopole equation perturbed by $\eta \in \Gamma(C_+^+\otimes_{\mathbb{R}}AdP_L)$ satisfying $\partial\!\!\!/_C\eta = 0$ is defined by

$$\begin{cases}\partial\!\!\!/_S\phi = \xi\phi,\\ \partial\!\!\!/_C\xi = -(\phi\otimes\phi^*)_0 - \sqrt{-1}\xi_{\#}\xi + \eta\end{cases} \tag{1}$$

for $(\phi,\xi) \in \Gamma(S^+\otimes_{\mathbb{C}}L) \oplus \Gamma(C_-^-\otimes_{\mathbb{R}}AdP_L)$. We denote by $\mathcal{S}_{MON,\eta}$ the space of all solutions of the perturbed Cliffordian monopole equation (1). We assume that $\mathrm{H}^i_{DR}(X) = 0$ for odd i and $1 + b^2 + \cdots + b^{2k-1} + b_+^{2k} > 1$, where $b^i = \dim \mathrm{H}^i_{\mathrm{DR}}(X)$ and $b_+^{2k} = \dim \mathrm{H}^{2k}_{\mathrm{DR},+}$, the dimension of the space of self-dual harmonic $2k$-forms. Our main result is the following:

Theorem 1.1. *For a generic perturbation $\eta \neq 0$, the space $\mathcal{S}_{MON,\eta}$ is a smooth oriented compact manifold of dimension*

$$\dim \mathcal{S}_{MON,\eta} = -\frac{\chi(X) + sign(X)}{2} + 2\langle ch(L)\hat{\mathcal{A}}(X), X\rangle + 1,$$

if $\mathcal{S}_{MON,\eta}$ exists. Furthermore, there is a one to one correspondence between the orientations of $\mathcal{S}_{MON,\eta}$ and the orientations of

$$\mathrm{H}^0_{DR}(X) \oplus \mathrm{H}^2_{DR}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{DR,+}(X),$$

where $\mathrm{H}^{2k}_{DR,+}(X)$ is the space of real harmonic self-dual $2k$-forms.

Theorem 1.1 follows from Theorem 3.2 , Corollary 3.2 and Lemma 4.3. Then we can define the smooth invariants $q_s(X, L), q'_s(X, L), q''_s(X, L)$ and the stable version invariant $\phi_s(X, L)$ of the pair (X, L) by using the moduli space $\mathcal{M}_{MON,\eta} = \mathcal{S}_{MON,\eta}/S^1$ and the non-linear Dirac operator $D + Q$.

Theorem 1.2. *Our integral invariants $q'_s(X, L), q''_s(X, L)$ and rational invariant $q_s(X, L)$ do not depend on the choice of Riemannian metric g on X, the choice of Hermitian metric h on L and the choice of generic perturbation η.*

Theorem 1.2 follows from Theorem 6.1 and Corollary 6.1.

The organization of this paper is the following. In Section 2, we give some preparations for our theory. We treat Clifford bundles, spinor bundles, Dirac operators, e.t.c.. In

Section 3, we define a perturbed Cliffordian monopole equation on a closed spin $4k$-manifold. In Section 4, we show that the space $\mathcal{S}$ of all solutions of a perturbed Cliffordian monopole equation has a structure of oriented finite dimensional manifold. In Section 5, we prove the compactness of $\mathcal{S}$. In Section 6, we define three invariants of X by using the index bundle of an elliptic family parameterized by the moduli space $\mathcal{M} = \mathcal{S}/G$ and the principal G-bundle structure of $\mathcal{S}$ over $\mathcal{M}$. In Section 7, we give some results for the existence problem of solutions of Cliffordian monopole equations.

2. Clifford bundles, Spin structures and Dirac operators

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of $\mathbb{R}^n$ with respect to the standard metric on $\mathbb{R}^n$. The (real) Clifford algebra $C(\mathbb{R}^n)$ of $\mathbb{R}^n$ is given by $C(\mathbb{R}^n) = \mathcal{T}(\mathbb{R}^n)/\mathcal{I}$, where $\mathcal{T}(\mathbb{R}^n) = \underset{i=0}{\oplus} \otimes^i \mathbb{R}^n$ is the tensor algebra and $\mathcal{I}$ is its two-sided ideal defined by $\mathcal{I} = \langle \{e_i e_j + e_j e_i + 2\delta_{ij}\} \rangle$. The Clifford algebra $C(\mathbb{R}^n)$ is a $\mathbb{Z}$-graded vector space $C^0(\mathbb{R}^n) \oplus C^1(\mathbb{R}^n) \oplus \cdots \oplus C^n(\mathbb{R}^n)$, where $C^i(\mathbb{R}^n)$ is the vector subspace of $C(\mathbb{R}^n)$ spanned by $\{e_{j_1} \cdots e_{j_i} \mid j_1 < \cdots < j_i\}$. We set $C^+(\mathbb{R}^n) = \underset{i=0}{\oplus} C^{2i}(\mathbb{R}^n)$ and $C^-(\mathbb{R}^n) = \underset{i=1}{\oplus} C^{2i-1}(\mathbb{R}^n)$. The Clifford algebra $C(\mathbb{R}^n)$ has a $\mathbb{Z}_2$-grading (super) algebra structure,

$$C(\mathbb{R}^n) = C^+(\mathbb{R}^n) \oplus C^-(\mathbb{R}^n),$$
$$C^\pm \cdot C^\pm \subset C^+, \quad C^\pm \cdot C^\mp \subset C^-.$$

For a finite ordered sequence $I = (i_1, \ldots, i_j)$ of distinct suffices $i_1, \ldots, i_j \in \{1, \ldots, n\}$, we use a short-hand notation $e_I = e_{i_1} \cdots e_{i_j}$, $L(I) = j$. Here, note that for $I = \emptyset$, we set $e_\emptyset = 1$ and $L(\emptyset) = 0$. We have a decomposition, $C^j(\mathbb{R}^n) = \underset{L(I)=j}{\oplus} \mathbb{R}\langle e_I \rangle$. Now, we assume that $n = 4k$ for $k \in \mathbb{Z}_{>0}$. The chirality element $\gamma = (\sqrt{-1})^{\left[\frac{n+1}{2}\right]} e_1 \cdots e_n = (\sqrt{-1})^{2k} e_1 \cdots e_{4k}$ is of $C(\mathbb{R}^n)$ satisfies $\gamma^2 = 1$.

Remark 1. Since the chirality element γ is independent of a choice of orthonormal basis of $\mathbb{R}^{4k}$, the $\mathbb{Z}$-grading structure of the $C(\mathbb{R}^{4k})$ is preserved under any special orthonormal transformation of $\mathbb{R}^{4k}$.

Definition 2.1. Let e_I be a base element of $C(\mathbb{R}^n)$. We call e_I of type A if ${e_I}^2 = -1$. We call e_I of type B if ${e_I}^2 = 1$.

Proposition 2.1. *A base e_I of $C(\mathbb{R}^n)$ is of type A* (or B) if and only if $T(I) := \sum_{l=0}^{L(I)} l$ is odd (or even). Moreover, when $L(I)$ is odd, e_I is of type A (or B) if and only if $e_I \gamma e_I = \gamma$ (or $-\gamma$). Furthermore, when $L(I)$ is even, e_I is of type A (or B) if and only if $e_I \gamma e_I = -\gamma$ (or γ).

The proof of Proposition 2.1 is omitted, because it is a simple calculation. We have the following table by using Proposition 2.1.

Table 2.1.

$L(I)$	0	1	2	3	4	$\cdots$	$4k-3$	$4k-2$	$4k-1$	$4k$
$T(I)=\sum_{l=0}^{L(I)} l$	0	1	3	6	10	$\cdots$	$(2k-1)$ $\times(4k-3)$	$(2k-1)$ $\times(4k-1)$	$2k$ $\times(4k-1)$	$2k$ $\times(4k+1)$
Type	B	A	A	B	B	$\cdots$	A	A	B	B

Next we consider useful vector subspaces of $C^-(\mathbb{R}^n)$.

Definition 2.2. We set

$$A(\mathbb{R}^n)=\sum_{j=1}^{k} C^{4j-3}(\mathbb{R}^n), \quad B(\mathbb{R}^n)=\sum_{j=1}^{k} C^{4j-1}(\mathbb{R}^n).$$

Table 2.1 shows that $A(\mathbb{R}^n)$ is spanned by the basis of type A and $B(\mathbb{R}^n)$ is spanned by the basis of type B. For the chirality element γ, we have

Proposition 2.2.

$$A(\mathbb{R}^n)\underset{\gamma}{\cong} B(\mathbb{R}^n), \quad C^-(\mathbb{R}^n)=A(\mathbb{R}^n)\oplus B(\mathbb{R}^n).$$

Definition 2.3. We set

$$C_+(\mathbb{R}^n)=\frac{1+\gamma}{2}C(\mathbb{R}^n), \quad C_-(\mathbb{R}^n)=\frac{1-\gamma}{2}C(\mathbb{R}^n).$$

The Clifford algebra has a decomposition:

$$C(\mathbb{R}^n)=C_+^+(\mathbb{R}^n)\oplus C_-^+(\mathbb{R}^n)\oplus C_+^-(\mathbb{R}^n)\oplus C_-^-(\mathbb{R}^n).$$

Then $C(\mathbb{R}^n)$ satisfies the following property:

Proposition 2.3. *The Clifford algebra $C(\mathbb{R}^n)$ satisfies the following property:*

$$\begin{aligned}
&C_+^+\cdot C_+^+\subset C_+^+, \quad C_-^+\cdot C_+^+=\{0\}, &&C_+^+\cdot C_+^-\subset C_+^-, \quad C_-^+\cdot C_+^-=\{0\},\\
&C_+^+\cdot C_-^+=\{0\}, \quad C_-^+\cdot C_-^+\subset C_-^+, &&C_+^+\cdot C_-^-=\{0\}, \quad C_-^+\cdot C_-^-\subset C_-^-,\\
&C_-^-\cdot C_+^+\subset C_-^-, \quad C_+^-\cdot C_+^+=\{0\}, &&C_-^-\cdot C_+^-\subset C_-^+, \quad C_+^-\cdot C_+^-=\{0\},\\
&C_-^-\cdot C_-^+=\{0\}, \quad C_+^-\cdot C_-^+\subset C_+^-, &&C_-^-\cdot C_-^-=\{0\}, \quad C_+^-\cdot C_-^-\subset C_+^+.
\end{aligned}$$

Proof. Let $a, b \in C_-^-$. We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$ and $b = \sum_{J:\text{type A}} \beta^I \frac{e_J - \gamma e_J}{\sqrt{2}}$. Then we have

$$
\begin{aligned}
ab &= \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}} \sum_{J:\text{type A}} \beta^J \frac{e_J - \gamma e_J}{\sqrt{2}} \\
&= \sum_{I,J:\text{type A}} \alpha^I \beta^J \frac{e_I e_J - \gamma e_I e_J - e_I \gamma e_J + \gamma e_I \gamma e_J}{2} \\
&= 0.
\end{aligned}
$$

Here we have used the fact that if $L(I)$ is odd then $e_I \gamma = -\gamma e_I$. Therefore we conclude that $C_-^- \cdot C_-^- = \{0\}$. We can show the rest of the cases by the same method as the above computation. □

Definition 2.4. The linear map $\#$ of $C_\pm^-(\mathbb{R}^n)$ to $C_\mp^-(\mathbb{R}^n)$ is defined by

$$
\#\left(\sum_{I:\text{type A}} \alpha^I \frac{e_I \pm \gamma e_I}{\sqrt{2}} \right) = \sum_{I:\text{type A}} \alpha^I \frac{e_I \mp \gamma e_I}{\sqrt{2}}
$$

for each $\sum_{I:\text{type A}} \alpha^I \frac{e_I \pm \gamma e_I}{\sqrt{2}} \in C_\pm(\mathbb{R}^n)$. We denote $\#(a)$ by $a_\#$. We also use the formal notation

$$
b_\sharp = \sum_{I,J:\text{odd, type A}} \beta^{I,J} \frac{e_I e_J \mp \gamma e_I e_J}{\sqrt{2}}
$$

for $b = \sum_{I,J:\text{odd, type A}} \beta^{I,J} \frac{e_I e_J \pm \gamma e_I e_J}{\sqrt{2}} \in C_\pm^+(\mathbb{R}^n)$.

Lemma 2.1. *For any $a \in C_\pm^-(\mathbb{R}^n)$ and $b \in C_\pm^-(\mathbb{R}^n)$, we have the formula:*

$$
(ab)_\# = a_\# b_\#.
$$

Proof. We prove the formula only in case $a \in C_-^-, b \in C_+^-$. In the other cases, we can prove them similarly. We write $a = \sum_{I:\text{A type}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$ and $b = \sum_{J:\text{A type}} \beta^J \frac{e_J + \gamma e_J}{\sqrt{2}}$. Then we have

$$
\begin{aligned}
(ab)_{\#} &= \left(\sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}} \sum_{I:\text{type A}} \beta^J \frac{e_J + \gamma e_J}{\sqrt{2}} \right)_{\#} \\
&= \left(\sum_{I,J:\text{type A}} \alpha^I \beta^J \frac{e_I e_J - \gamma e_I e_J + e_I \gamma e_J - \gamma e_I \gamma e_J}{2} \right)_{\#} \\
&= \left(\sum_{I,J:\text{type A}} \alpha^I \beta^J (e_I e_J - \gamma e_I e_J) \right)_{\#} \\
&= \sum_{I,J:\text{type A}} \alpha^I \beta^J (e_I e_J + \gamma e_I e_J),
\end{aligned}
$$

and

$$
\begin{aligned}
a_{\#} b_{\#} &= \sum_{I:\text{type A}} \alpha^I \frac{e_I + \gamma e_I}{\sqrt{2}} \sum_{I:\text{type A}} \beta^J \frac{e_J - \gamma e_J}{\sqrt{2}} \\
&= \sum_{I,J:\text{type A}} \alpha^I \beta^J \frac{e_I e_J + \gamma e_I e_J - e_I \gamma e_J - \gamma e_I \gamma e_J}{2} \\
&= \sum_{I,J:\text{type A}} \alpha^I \beta^J (e_I e_J + \gamma e_I e_J).
\end{aligned}
$$

Therefore we have the lemma. □

Definition 2.5. For each $a \in C^-_-(\mathbb{R}^n)$, we define $\hat{\delta}(a) \in C^+_+(\mathbb{R}^n)$ as follows: We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$. Then $\hat{\delta}(a)$ is given by

$$
\hat{\delta}(a) = \sum_{\substack{I \neq J:\text{type A} \\ e_I e_J = e_J e_I}} \alpha^I \alpha^J (e_I e_J + \gamma e_I e_J).
$$

Lemma 2.2. *For any $a \in C^-_-(\mathbb{R}^n)$, we have the following formulas:*

$$
a_{\#} a = -(1+\gamma)|a|^2 + \hat{\delta}(a), \quad a a_{\#} = -(1-\gamma)|a|^2 + \hat{\delta}(a)_{\#}.
$$

Proof. We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$. Then we have

$$
\begin{aligned}
a_{\#} a &= \sum_{I:\text{type A}} \alpha^I \frac{e_I + \gamma e_I}{\sqrt{2}} \sum_{J:\text{type A}} \alpha^J \frac{e_J - \gamma e_J}{\sqrt{2}} \\
&= \sum_{I,J:\text{type A}} \alpha^I \alpha^J (e_I e_J + \gamma e_I e_J).
\end{aligned}
$$

Since $e_I^2 = -1$ for each I of type A, we have

$$\begin{aligned} a_{\#}a &= \sum_{I(=J):\text{type A}} \alpha^I \alpha^I (e_I e_J - \gamma e_I e_I) + \sum_{I \neq J:\text{type A}} \alpha^I \alpha^J (e_I e_J - \gamma e_I e_J) \\ &= \sum_{I(=J):\text{type A}} \alpha^I \alpha^I (e_I e_J - \gamma e_I e_I) + \sum_{\substack{I \neq J:\text{type A} \\ e_I e_J = e_J e_I}} \alpha^I \alpha^J (e_I e_J - \gamma e_I e_J) \\ &= \sum_{I:\text{type A}} (\alpha^I)^2 (-1-\gamma) + \hat{\delta}(a) \\ &= -(1+\gamma)|a|^2 + \hat{\delta}(a). \end{aligned}$$

In similar way, we have the second formula. □

Remark 2. Since $e_I e_J e_I e_J = e_I e_J e_J e_I = (-1)^2 = 1$, we conclude that $\hat{\delta}(a)$ is of type B. Therefore $\hat{\delta}(a) \in \oplus_{i=2}^{k-1} C^{4i}(\mathbb{R}^n)$.

Remark 3. On $\mathbb{R}^4$, $A(\mathbb{R}^4)$ is spanned by $\{e_1, e_2, e_3, e_4\}$ such that $e_i e_j = -e_j e_i$. Thus we have $\hat{\delta}(a) = 0$ for any $a \in C_-^-(\mathbb{R}^4)$. Therefore we have $a_{\#}a = -(1+\gamma)|a|^2$.

Corollary 2.1. *For any $a \in C_-^-(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$, we have the following formulas:*

$$a_{\#}a = (1+\gamma)|a|^2 + \hat{\delta}(a), \quad a a_{\#} = (1-\gamma)|a|^2 + \hat{\delta}(a)_{\#}.$$

Lemma 2.3. *For any $a, b \in C_-^-(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$, the inner product $\langle a, b \rangle$ is equal to the coefficient of $1+\gamma$ of $a_{\#}b$.*

Proof. We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$ and $b = \sum_{J:\text{type A}} \beta^J \frac{e_J - \gamma e_J}{\sqrt{2}}$. Then

$$\langle a, b \rangle = \sum_{I:\text{type A}} \overline{\alpha^I} \beta^I = -\sum_{I:\text{type A}} \alpha^I \beta^I.$$

In another way,

$$\begin{aligned} a_{\#}b &= \sum_{I:\text{type A}} \alpha^I \frac{e_I + \gamma e_I}{\sqrt{2}} \sum_{J:\text{type A}} \beta^J \frac{e_J - \gamma e_J}{\sqrt{2}} \\ &= \sum_{I,J:\text{type A}} \alpha^I \beta^J (e_I e_J + \gamma e_I e_J) \\ &= -\sum_{I:\text{type A}} \alpha^I \beta^I (1+\gamma) + \sum_{I \neq J:\text{type A}} \alpha^I \beta^J (e_I e_J + \gamma e_I e_J). \end{aligned}$$

Therefore we conclude the assertion of the lemma. □

Lemma 2.4. *For $a \in C_-^-(\mathbb{R}^n)\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$, the inequality*

$$0 \leq \langle a\hat{\delta}(a), a\rangle \leq 4|a|^4$$

holds.

Proof. We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$. For simplicity, we denote the coefficient of $1+\gamma$ of any $b \in C_+^+\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$ by $[b]$. Since

$$aa_{\#}a = (1-\gamma)|a|^2 a + a\hat{\delta}(a) = (1-\gamma)|a|^2 a + \hat{\delta}(a)_{\#}a,$$

we have by Lemma 2.3

$$\begin{aligned} \langle a\hat{\delta}(a), a\rangle &= \langle \hat{\delta}(a)_{\#}a, a\rangle \\ &= \left[\hat{\delta}(a)a_{\#}a\right] \\ &= \left[\hat{\delta}(a)\{(1+\gamma)|a|^2 + \hat{\delta}(a)\}\right] \\ &= \left[\hat{\delta}(a)\hat{\delta}(a)\right]. \end{aligned}$$

Hence we compute $\hat{\delta}(a)\hat{\delta}(a)$. Indeed,

$$\begin{aligned} \hat{\delta}(a)\hat{\delta}(a) &= \sum_{\substack{I\neq J:\text{type A} \\ e_Ie_J=e_Je_I}} \alpha^I\alpha^J(e_Ie_J + \gamma e_Ie_J) \sum_{\substack{K\neq L:\text{type A} \\ e_Ke_L=e_Le_K}} \alpha^K\alpha^L(e_Ke_L + \gamma e_Le_K) \\ &= 2\sum_{\substack{I\neq J, K\neq L:\text{type A} \\ e_Ie_J=e_Je_I \\ e_Ke_L=e_Le_K}} \alpha^I\alpha^J\alpha^K\alpha^L(e_Ie_Je_Ke_L + \gamma e_Ie_Je_Ke_L). \end{aligned}$$

Thus we have

$$\begin{aligned} [\hat{\delta}(a)\hat{\delta}(a)] &= 2\sum_{\substack{I\neq J:\text{type A} \\ e_Ie_J=e_Je_I}} \alpha^I\alpha^J\alpha^I\alpha^J(e_Ie_Je_Ie_J + \gamma e_Ie_Je_Ie_J) \\ &\quad + 2\sum_{\substack{I\neq J:\text{type A} \\ e_Ie_J=e_Je_I}} \alpha^I\alpha^J\alpha^J\alpha^I(e_Ie_Je_Je_I + \gamma e_Ie_Je_Je_I) \\ &= 4\sum_{\substack{I\neq J:\text{type A} \\ e_Ie_J=e_Je_I}} (\alpha^I)^2(\alpha^J)^2. \end{aligned}$$

On the other hand, the inner product $\langle aa_\# a, a\rangle = 2|a|^2 + \langle\hat{\delta}(a)_\# a, a\rangle$ is re-computed as follows:

$$
\begin{aligned}
\langle aa_\# a, a\rangle &= \left\langle \sqrt{2}\sum_{I,J,K:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&= -2\left\langle \sum_{I=J,K:\text{type A}} (\alpha^I)^2\alpha^K\frac{e_K-\gamma e_K}{\sqrt{2}}, \sum_{L:\text{type A}} \frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&\quad - 2\left\langle \sum_{J=K,I:\text{type A}} (\alpha^J)^2\alpha^I\frac{e_I-\gamma e_I}{\sqrt{2}}, \sum_{L:\text{type A}} \frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&\quad - 2\left\langle \sum_{K=I,J:\text{type A}} (\alpha^K)^2\alpha^J\frac{e_J-\gamma e_J}{\sqrt{2}}, \sum_{L:\text{type A}} \frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&\quad + 4\sqrt{2}\left\langle \sum_{I(=J=K):\text{type A}} (\alpha^I)^3\frac{e_I-\gamma e_I}{\sqrt{2}}, \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&\quad + \sqrt{2}\left\langle \sum_{I\neq J,J\neq K,K\neq I:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&= 6|a|^4 - 4\sum_{I:\text{type A}}(\alpha^I)^4 \\
&\quad + \sqrt{2}\left\langle \sum_{I\neq J,J\neq K,K\neq I:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&= 6|a|^2 - 4\left\{\sum_{I:\text{type A}}(\alpha^I)^2\sum_{J:\text{type A}}(\alpha^J)^2 - \sum_{I\neq J}(\alpha^I)^2(\alpha^J)^2\right\} \\
&\quad + \sqrt{2}\left\langle \sum_{I\neq J,J\neq K,K\neq I:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&= 6|a|^4 - 4|a|^4 + 4\sum_{I\neq J:\text{type A}}(\alpha^I)^2(\alpha^J)^2 \\
&\quad + \sqrt{2}\left\langle \sum_{I\neq J,J\neq K,K\neq I:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle \\
&= 2|a|^4 + 4\sum_{I\neq J:\text{type A}}(\alpha^I)^2(\alpha^J)^2 \\
&\quad + \sqrt{2}\left\langle \sum_{I\neq J,J\neq K,K\neq I:\text{type A}} \alpha^I\alpha^J\alpha^K(e_Ie_Je_K - \gamma e_Ie_Je_K), \sum_{L:\text{type A}} \alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle.
\end{aligned}
$$

Therefore we have

$$\begin{aligned}\langle\hat{\delta}(a)_{\#}a,a\rangle &= 4\sum_{\substack{I\neq J:\text{type A}\\ e_Ie_J=e_Je_I}}(\alpha^I)^2(\alpha^J)^2\\ &= 4\sum_{I\neq J:\text{type A}}(\alpha^I)^2(\alpha^J)^2\\ &\quad -\sqrt{2}\left\langle\sum_{I\neq J,J\neq K,K\neq I:\text{type A}}\alpha^I\alpha^J\alpha^K(e_Ie_Je_K-\gamma e_Ie_Je_K),\sum_{L:\text{type A}}\alpha^L\frac{e_L-\gamma e_L}{\sqrt{2}}\right\rangle.\end{aligned}$$

Therefore we conclude that

$$0\leq\langle\hat{\delta}(a)_{\#}a,a\rangle\leq 4|a|^4.$$

□

Corollary 2.2. *For any* $a\in C^-_-(\mathbb{R}^n)\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$, *we have*

$$2|a|^4\leq\langle aa_{\#}a,a\rangle\leq 6|a|^4.$$

Proof. This corollary immediately follows from Lemma 2.4. □

Let X be a smooth closed oriented manifold of dimension $n=4k$ with a Riemannian metric g. We denote by F_X the orthonormal frame bundle of TX. F_X is a principal $\mathrm{SO}(n)$-bundle. A special orthogonal transformation of $\mathbb{R}^n$ induces an automorphism of $C(\mathbb{R}^n)$ preserving the norm. Thus we have a representation $cl\colon\mathrm{SO}(n)\longrightarrow\mathrm{Aut}(C(\mathbb{R}^n))$.

Definition 2.6. The Clifford bundle $C(X)$ over X is defined by

$$C(X)=F_X\times_{cl}C(\mathbb{R}^n).$$

Furthermore, two sub-vector bundles $A(X)$ and $B(X)$ of $C(X)$ are defined by

$$A(X)=F_X\times_{cl}A(\mathbb{R}^n),$$
$$B(X)=F_X\times_{cl}B(\mathbb{R}^n).$$

We simply write $C=C(X),A=A(X)$ and $B=B(X)$. The Clifford bundle has natural $\mathbb{Z}_2$-graded algebra structure,

$$C(X)=C^+(X)\oplus C^-(X),$$

where both $C^+(X)$ and $C^-(X)$ are $C^+(X)$-modules and $C^\pm\cdot C^\pm\subset C^+,\quad C^\pm\cdot C^\mp\subset C^-$. $C^-(X)$ decomposes into

$$C^-(X)=A(X)\oplus B(X)$$

as a vector bundle. We denote by $C^\times(\mathbb{R}^n)$ the set of all units in $C(\mathbb{R}^n)$. We recall that $\mathrm{Pin}(n)=\langle\{v\in\mathbb{R}^n\mid|v|=1\}\rangle\subset C^\times(\mathbb{R}^n)$ and $\mathrm{Spin}(n)=\mathrm{Pin}(n)\cap C^+(\mathbb{R}^n)$.

Definition 2.7. A smooth oriented manifold X is called a spin manifold if there exists a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}(n)}$ over X and a double covering map ρ of $P_{\mathrm{Spin}(n)}$ to F_X such that the diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}(n)} & \xrightarrow{\rho} & F_X \\ {\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\pi}\downarrow \\ X & = & X \end{array}$$

is commutative, where $\tilde{\pi}$ and π are the projections to X. The isomorphism class of $P_{\mathrm{Spin}(n)}$ is called a spin structure on X.

X is spin if and only if $w_2(X) = 0$. For $n \geq 3$, $\mathrm{Spin}(n)$ is a universal double covering of $\mathrm{SO}(n)$. The space $\mathrm{Spin}(X)$ of all spin structures on X has an affine space structure $\mathrm{Spin}(X) = s + \mathrm{H}^1(X;\mathbb{Z}_2)$ where $s \in \mathrm{Spin}(X)$. The complex spin representation of $\mathrm{Spin}(n)$ is an irreducible ($\mathbb{Z}_2$-grading) unitary representation

$$\Delta\colon \mathrm{Spin}(n) \longrightarrow \mathrm{GL}(M;\mathbb{C}),$$

where

$$M = \begin{cases} \mathbb{C}^{2^{4k}} & n = 8k, \quad k = 1, 2, \dots \\ \mathbb{H}^{2^{2k+1}} & n = 8k+4, \quad k = 0, 1, \dots \end{cases},$$

because the maximal commuting subalgebra K_n for real representations of $C(\mathbb{R}^n)$ is $\mathbb{C}$ if $n \equiv 0 \pmod 8$, $\mathbb{H}$ if $n \equiv 4 \pmod 8$ (,see Atiyah-Bott-Shapiro [2] and Lawson-Michelsohn [20]). If n is even, then the $\mathrm{Spin}(n)$-module M has the $\mathbb{Z}_2$-grading $C(\mathbb{R}^n)$-module structure:

$$C^{\pm} \cdot M^{\pm} \subset M^{+}, \quad C^{\pm} \cdot M^{\mp} \subset M^{-}.$$

In particular, if $n = 4k$, then $M^{\pm}$ is the ± 1-eigenspace of the chirality operator γ. Therefore M has the following property

$$\begin{array}{ll} C^{+}_{+} \cdot M^{+} \subset M^{+}, \quad C^{-}_{+} \cdot M^{+} = \{0\}, & \quad C^{+}_{-} \cdot M^{+} = \{0\}, \quad C^{-}_{-} \cdot M^{+} \subset M^{-} \\ C^{+}_{+} \cdot M^{-} = \{0\}, \quad C^{-}_{+} \cdot M^{-} \subset M^{+}, & \quad C^{+}_{-} \cdot M^{-} \subset M^{-}, \quad C^{-}_{-} \cdot M^{-} = \{0\}. \end{array}$$

Lemma 2.5. *For $a \in C^{-}_{-}(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$ and $s \in M^{+}$, we have*

$$\langle a_{\#} a s, s \rangle_{M^{+}} = \langle a s, a s \rangle_{M^{-}}.$$

Proof. We write $a = \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}}$. Then we have

$$\begin{aligned}
\langle a_{\#} a s, s\rangle_{M^+} &= \left\langle \sum_{I,J:\text{type A}} \alpha^I \alpha^J (e_I e_J + \gamma e_I e_J) s, s \right\rangle_{M^+} \\
&= 2 \sum_{I,J:\text{type A}} \overline{\alpha^I \alpha^J} \langle e_I e_J s, s\rangle_{M^+} \\
&= -2 \sum_{I,J:\text{type A}} \overline{\alpha^I \alpha^J} \langle e_J s, e_I s\rangle_{M^-} \\
&= 2 \sum_{I,J:\text{type A}} \alpha^I \overline{\alpha^J} \langle e_J s, e_I s\rangle_{M^-} \\
&= 2 \left\langle \sum_{J:\text{type A}} \alpha^J e_J s, \sum_{I:\text{type A}} \alpha^I e_I s \right\rangle_{M^-} \\
&= \left\langle \sum_{J:\text{type A}} \alpha^J \frac{e_J - \gamma e_J}{\sqrt{2}} s, \sum_{I:\text{type A}} \alpha^I \frac{e_I - \gamma e_I}{\sqrt{2}} s \right\rangle_{M^-} \\
&= \langle a s, a s\rangle_{M^-}.
\end{aligned}$$

□

Lemma 2.6. *For $a \in C_-^-(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$ and $s \in M^+$, we have*

$$\langle \hat{\delta}(a) s, s\rangle_{M^+} = |as|^2 - 2|a|^2 |s|^2.$$

Proof. By Lemma 2.5, we have

$$\begin{aligned}
|as|^2 &= \langle as, as\rangle_{M^-} \\
&= \langle a_{\#} a s, s\rangle_{M^+} \\
&= 2|a|^2|s|^2 + \langle \hat{\delta}(a) s, s\rangle_{M^+}.
\end{aligned}$$

□

Corollary 2.3. *For $a \in C_-^-(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$ and $s \in M^+$, we have the following inequality:*

$$-2|a|^2|s|^2 \leq \langle \hat{\delta}(a) s, s\rangle_{M^+} \leq 0.$$

Proof. Since $0 \leq |as|^2 \leq 2|a|^2|s|^2$, we conclude the assertion of the corollary. □

Corollary 2.4. *For $a \in C_-^-(\mathbb{R}^n) \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R}$ and $s \in M^+$, we have the following inequality:*

$$0 \leq \langle a_{\#} a s, s\rangle_{M^+} \leq 2|a|^2|s|^2.$$

Proof. This corollary immediately follows from the above corollary. □

Definition 2.8. When X has a spin structure, we define the (complex) spinor bundle by

$$S = P_{\mathrm{Spin}(n)} \times_{\Delta} M.$$

There are well-known facts;

$$C(X) \cong P_{\mathrm{Spin}(n)} \times_{Ad} C(\mathbb{R}^n), \quad \mathrm{End}(S) \cong C(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

The spinor bundle S has a $\mathbb{Z}_2$-grading $C(X)$-module structure,

$$S = S^+ \oplus S^-,$$

where S^+, S^- are $C^+(X)$-module such that

$$C^{\pm} \cdot S^{\pm} \subset S^+, \quad C^{\pm} \cdot S^{\mp} \subset S^-.$$

From now on, we assume that X is a closed spin manifold of dimension $n = 4k$.

Definition 2.9. The chirality operator γ is given by the local expression:

$$\begin{aligned} \gamma &= (\sqrt{-1})^{[\frac{n+1}{2}]} e_1 e_2 \cdots e_n \\ &= (\sqrt{-1})^{2k} e_1 e_2 \cdots e_{4k-1} e_{4k}, \end{aligned}$$

where $(e_1, \ldots, e_{4k})$ is a local frame of $C^1 = TX$. The chirality operator γ is independent of the choice of a local frame of TX. $\gamma \in \Gamma(C^{4k}(X)) \subset \Gamma(C(X))$.

In general, $\gamma \in \Gamma(C(X) \otimes_{\mathbb{R}} \mathbb{C})$ holds. The chirality operator induces an isomorphism of $C(X)$ satisfying $\gamma^2 = 1$. We write the ± 1-eigen spaces of γ by

$$C_{\pm}(X) = \frac{1 \pm \gamma}{2} C(X).$$

We have an orthogonal decomposition of $C(X)$ as a vector bundle:

$$C(X) = C_+^+(X) \oplus C_-^+(X) \oplus C_+^-(X) \oplus C_-^-(X).$$

We can regard the halves of the spinor bundle $S^{\pm}$ as the ± 1-eigen space of γ respectively (,see Berline-Getzler-Vergne [8]). Therefore the spinor bundle $S(X)$ has the following property:

$$\begin{aligned} &C_+^+ \cdot S^+ \subset S^+, \quad C_+^- \cdot S^+ = \{0\}, \qquad C_-^+ \cdot S^+ = \{0\}, \quad C_-^- \cdot S^+ \subset S^- \\ &C_+^+ \cdot S^- = \{0\}, \quad C_+^- \cdot S^- \subset S^+, \qquad C_-^+ \cdot S^- \subset S^-, \quad C_-^- \cdot S^- = \{0\}. \end{aligned}$$

A $C(X)$-module bundle $\mathcal{E}$ is a $\mathbb{Z}_2$-grading vector bundle over X such that $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ where $\mathcal{E}^{\pm}$ are $C^+(X)$-modules and $C^{\pm} \cdot \mathcal{E}^{\pm} \subset \mathcal{E}^+, \quad C^{\pm} \cdot \mathcal{E}^{\mp} \subset \mathcal{E}^-$. We can consider the Cliffordian multiplication

$$\begin{array}{ccc} \Gamma(C) \times \Gamma(\mathcal{E}) & \longrightarrow & \Gamma(\mathcal{E}) \\ \xi \times s & \longmapsto & \xi s \end{array}.$$

Definition 2.10. Let ∇ be a connection on a $C(X)$-module $\mathcal{E}$ compatible with metric on $\mathcal{E}$. The Dirac operator $\partial\!\!\!/_\nabla$ for ∇ is defined as the composition of maps:

$$\partial\!\!\!/_\nabla\colon \Gamma(\mathcal{E}) \xrightarrow{\ \nabla\ } \Gamma(T^*(X)\otimes\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}),$$

where $T^*X = C^1(X)\subset C(X)$ and $\Gamma(T^*X\otimes\mathcal{E})\longrightarrow\Gamma(\mathcal{E})$ is the Clifford multiplication.

The Dirac operator $\partial\!\!\!/_\nabla$ is a 1st order linear elliptic operator on $\Gamma(\mathcal{E})$. $\partial\!\!\!/_\nabla$ has an odd parity with respect to $\mathbb{Z}_2$-grading on $\mathcal{E}$. The Clifford bundle C and the spinor bundle S are both $C(X)$-modules over X. The Levi-Civita connection ∇_X on TX with respect to the Riemannian metric g induces connections on C and S. We denote by $\partial\!\!\!/_S$ and $\partial\!\!\!/_C$ the Dirac operators on $\Gamma(S)$ and $\Gamma(C)$ for the connections induced by ∇_X on C and S respectively. In particular, we can consider the half of the Dirac operator $\partial\!\!\!/_S\colon \Gamma(S^+)\longrightarrow\Gamma(S^-)$ and the quarter of the Dirac operator $\partial\!\!\!/_C\colon \Gamma(C^-_-)\longrightarrow\Gamma(C^+_+)$, because the Dirac operator $\partial\!\!\!/_C$ and the chirality operator γ anti-commute to each other , i.e., $\partial\!\!\!/_C\gamma = -\gamma\partial\!\!\!/_C$. These are also 1-st order linear elliptic operators. Let E be a vector bundle over X. Then we can consider the extensions of the Dirac operators, $\partial\!\!\!/_S\colon \Gamma(S\otimes E)\longrightarrow\Gamma(S\otimes E)$ and $\partial\!\!\!/_C\colon \Gamma(C\otimes AdP_E)\longrightarrow\Gamma(C\otimes AdP_E)$ with respect to a connection on E compatible with a Hermitian metric on E, where P_E is principal $\mathrm{SO}(r)$-bundle associated with E and $r=\mathrm{rank}E$. We call them the twisted Dirac operators.

We now study the quarter of the Dirac operator $\partial\!\!\!/_C\colon \Gamma(C^-_-)\longrightarrow\Gamma(C^+_+)$. We call the isomorphism as vector bundles

$$\sigma\colon C(X)\xrightarrow{\ \cong\ }\Lambda^*T^*X$$

the symbol map. Let $c=\sigma^{-1}$. We call c the quantization map. The Hodge's $*$-operator on Λ^*T^*X induces a linear isomorphism $*\colon \Omega^*(X)\longrightarrow\Omega^*(X)$ satisfying $*^2=1$. Let

$$\Omega^*_\pm = \frac{1\pm *}{2}\Omega^*$$

be the ± 1-eigen spaces of $*$ respectively. Let Ω^+ (resp. Ω^-) be the space of even (resp. odd) differential forms. Then Ω^* has a decomposition

$$\Omega^* = \Omega^+_+\oplus\Omega^+_-\oplus\Omega^-_+\oplus\Omega^-_-.$$

The symbol map σ induces the isomorphisms $\Omega^\alpha_\beta\cong\Gamma(C^\alpha_\beta)$, where $\alpha,\beta\in\{\pm\}$. Let $d\colon\Omega^*\longrightarrow\Omega^*$ be the exterior derivative and d^* the formal adjoint operator of d.

Lemma 2.7. (**Gauss-Bonnet-Chern**) *The operator $(d+d^*)\colon\Omega^+\longrightarrow\Omega^-$ is a 1st order elliptic operator and*

$$\mathrm{index}_{\mathbb{R}}\,(d+d^*) = \frac{1}{(2\pi)^{2k}}\int_X Pf(-R) = \chi(X).$$

Here R is the Riemannian curvature form of ∇_X, $Pf(-R)=\det^{\frac{1}{2}}(-R)=\int_{\mathrm{Berezin}}\exp(-R)$ and $\chi(X)$ is the Euler number of X.

On the other hand

Lemma 2.8. (**Hirzebruch**) *The operator* $(d+d^*)\colon \Omega_+ \longrightarrow \Omega_-$ *is a 1st order elliptic operator and*

$$\mathrm{index}_{\mathbb{R}}\,(d+d^*) = \frac{1}{(\sqrt{-1}\pi)^{2k}}\int_X L(X) = sign(X).$$

Here $L(X) = \det^{\frac{1}{2}}\left(\frac{R/2}{\tanh(R/2)}\right)$ *and* $sign(X)$ *is the signature of* X.

The following diagram

$$\begin{array}{ccc} \Gamma(C_-^-) & \xrightarrow{\partial\!\!\!/_C} & \Gamma(C_+^+) \\ \sigma\downarrow & & \downarrow\sigma \\ \Omega_+^+ & \xrightarrow{d+d^*} & \Omega_-^- \end{array}$$

is commutative, since $\partial\!\!\!/_C\gamma = -\gamma\partial\!\!\!/_C$. Therefore we have

Lemma 2.9. *The quarter of the Dirac operator* $\partial\!\!\!/_C\colon \Gamma(C_-^-) \longrightarrow \Gamma(C_+^+)$ *has*

$$\begin{aligned}\mathrm{index}_{\mathbb{R}}\,\partial\!\!\!/_C &= -\frac{1}{2(2\pi)^{2k}}\int_X \big(Pf(-R) + (-1)^k 2^k L(X)\big)\\ &= -\frac{\chi(X) + sign(X)}{2}.\end{aligned}$$

See Atiyah-Hitchin-Singer [4] and Atiyah-Savilian [5] in 4-dimensional case. For the index problem, the operator $(d+d^*)\colon \Omega_+^+ \longrightarrow \Omega_-^-$ is equivalent to the extended AHS-complex

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2k-1} \xrightarrow{d^+} \Omega_+^{2k} \longrightarrow 0,$$

where $d^+ = p_+ \circ d$ and $p_+\colon \Omega^{2k} \longrightarrow \Omega_+^{2k}$ is the projection.

In the rest of this section, we explain the relation between the # operator and the Dirac operator $\partial\!\!\!/$. For any section $\xi \in \Gamma(C_-^-)$, $\xi_\# \in \Gamma(C_+^-)$ globally exists. Also we can consider the products $\xi_\#\xi \in \Gamma(C_+^+)$ and $\xi\xi_\# \in \Gamma(C_-^-)$ as global sections.

Lemma 2.10. *For a pair* $(\phi, \xi) \in \Gamma(S^+) \oplus \Gamma(C_-^-)$ *satisfying the relation* $\partial\!\!\!/_S\phi = \xi\phi$, *we have the following formula:*

$${\partial\!\!\!/_S}^2\phi = (\partial\!\!\!/_C\xi)\phi + \xi_\#\xi\phi.$$

Proof. We choose a normal coordinate $(U, \{x_1, x_2, \ldots, x_n\})$ around a point $x \in X$. We assume that $(e_1 = c(dx^1), e_2 = c(dx^2), \ldots e_n = c(dx^n))$ is an orthonormal basis on U which satisfies the condition $\nabla_{e_i} e_j = 0$ for all $i, j \in \{1, 2, \ldots, n\}$. We can write $\xi = \sum_{I:\mathrm{odd}} \alpha^I e_I$. Then $\alpha^I = -\alpha^J$ for $e_I = \gamma e_J$. We have

$$\begin{aligned}
\partial\!\!\!/_S{}^2 \phi &= \partial\!\!\!/_S(\xi\phi) \\
&= (\partial\!\!\!/_C \xi)\phi + c(\xi\nabla\phi) \\
&= (\partial\!\!\!/_C \xi)\phi + \sum_{i=1}^{n} e_i \sum_{I:\mathrm{odd}} \alpha^I e_I \nabla_i \phi \\
&= (\partial\!\!\!/_C \xi)\phi - \sum_{i=1}^{n} \sum_{I:\mathrm{odd}} \alpha^I e_I e_i \nabla_i \phi + 2\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi \\
&= (\partial\!\!\!/_C \xi)\phi - \xi(\partial\!\!\!/_S \phi) + 2\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi.
\end{aligned}$$

Since $C_-^- S^+ = \{0\}$, we have $\xi(\partial\!\!\!/_S \phi) = 0$. Since $\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi \in \Gamma(S^+)$, we have

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi &= \gamma \sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi \\
&= \sum_{i=1}^{n} \sum_{i\in I} \alpha^I (\gamma e_I) e_i \nabla_i \phi.
\end{aligned}$$

Therefore we have

$$2\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi = \sum_{i=1}^{n} \sum_{i\in I} \alpha^I (e_I + \gamma e_I) e_i \nabla_i \phi.$$

Since for a fixed index $i \in \{1, 2, \ldots, n\}$, we can write

$$\sum_{i\in I} \alpha^I (e_I + \gamma e_I) = \xi_\#,$$

we have

$$\begin{aligned}
2\sum_{i=1}^{n} \sum_{i\in I} \alpha^I e_I e_i \nabla_i \phi &= \xi_\#(\partial\!\!\!/_S \phi) \\
&= \xi_\# \xi \phi.
\end{aligned}$$

Therefore we have the lemma. □

Remark 4. If X is 4-dimensional, then, by Remark 3, Lemma 2.10 implies that $\partial\!\!\!/_S{}^2 \phi = (\partial\!\!\!/_C \xi)\phi - 2|\xi|^2 \phi$.

Lemma 2.11. *For any $\xi \in \Gamma(C^-_\pm)$, we have the local formula:*

$$\partial\!\!\!/_C \xi_\# = (\partial\!\!\!/_C \xi)_\#.$$

(Note that this formula essencially is a formal expression, because $\sharp\colon C^+_\pm \longrightarrow C^+_\mp$ do not globally exist. See Definition 2.4)

Proof. This lemma is easily obtained by direct calculations. □

Lemma 2.12. *For any $\phi \in \Gamma(S^+ \otimes_{\mathbb{R}} \mathbb{C})$ and $\xi \in \Gamma(C^-_- \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R})$, we obtain*

$$\langle \xi_\# \xi \phi, \phi \rangle_{S^+ \otimes_{\mathbb{R}} \mathbb{C}} = 2|\xi|^2 |\phi|^2 + \langle \hat{\delta}(\xi)\phi, \phi \rangle_{S^+ \otimes_{\mathbb{R}} \mathbb{C}},$$

and

$$-2|\xi|^2|\phi|^2 \leq \langle \hat{\delta}(\xi)\phi, \phi \rangle_{S^+ \otimes_{\mathbb{R}} \mathbb{C}} \leq 0.$$

Proof. This lemma immediately follows from Corollaries 2.3 and 2.4. □

Lemma 2.13. *For any $\xi \in \Gamma(C^-_- \otimes_{\mathbb{R}} \sqrt{-1}\mathbb{R})$, we have the following inequality:*

$$2|\xi|^4 \leq \langle \xi \xi_\# \xi, \xi \rangle \leq 6|\xi|^4.$$

Proof. This lemma immediately follows from Corollary 2.2. □

3. A Cliffordian monopole equation on a spin $4k$-manifold

Let X be a smooth connected closed spin manifold of dimension $n = 4k$ with a Riemannian metric g. Let us assume that X is connected. Let L be a complex line bundle over X with a Hermitian metric h. Let P_L be a principal U(1)-bundle over X such that $L = P_L \times_{\mathrm{U}(1)} \mathbb{C}$ and $AdP_L = P_L \times_{Ad} \mathfrak{u}(1)$. We have that $AdP_L = X \times \sqrt{-1}\mathbb{R}$. Let A be a connection on L compatible with the metric h. We only treat the (twisted) Dirac operators $\partial\!\!\!/_S$ and $\partial\!\!\!/_C$ for the Levi-Civita connection on X with respect to g and Hermitian connection A with respect to h. We assume the following for the technical reason of the compactness of the solutions.

Assumption 3.1. $\mathrm{H}^i_{DR}(X) = 0$ *for odd i and* $1 + b^2 + \cdots + b^{2k-1} + b^{2k}_+ > 1$.

We remark that the set of all $4k$-manifolds satisfying the assumption is closed with respect to the operations $\times$ (direct product) and $\sharp$ (connected sum). We use the following shorthand notation:

$$C_L = C \otimes_{\mathbb{R}} AdP_L = \sqrt{-1}C,$$
$$S_L = S \otimes_{\mathbb{C}} L.$$

Definition 3.1. For $\xi \in \Gamma(C_{L_-}^-)$, the Cliffordian ASD-equation is given by

$$\partial\!\!\!/_C \xi = -\sqrt{-1}\xi_\# \xi,$$

where $\partial\!\!\!/_C \colon \Gamma(C_{L_-}^-) \longrightarrow \Gamma(C_{L_+}^+)$ is the quarter of the twisted Dirac operator.

We define $D_1 = \partial\!\!\!/_C \colon \Gamma(C_{L_-}^-) \longrightarrow \Gamma(C_{L_+}^+)$. We define $Q_1 \colon \Gamma(C_{L_-}^-) \longrightarrow \Gamma(C_{L_-}^-)$ by

$$Q_1(\xi) = \sqrt{-1}\xi_\# \xi.$$

Then we can write the Cliffordian ASD equation as

$$D_1 + Q_1 \colon \Gamma(C_{L_-}^-) \longrightarrow \Gamma(C_{L_+}^+).$$

For a non-negative integer l, the Sobolev space $L_l^2(F)$ for a Hermitian vector bundle F over X is the Banach completion of the space $\Gamma(F)$ of all sections of F with respect to the L_l^2-norm

$$\|f\|_{L_l^2} = \int_X \sum_{i=0}^{l} |\nabla_F^{(i)} f|^2 dvol$$

for each $f \in \Gamma(F)$, where ∇_F is a unitary connection on F and $\nabla_F^{(i)} = \nabla_F \circ \cdots \circ \nabla_F$ (i-times).

We denote by H the space $\left\{\eta \in L_2^l(C_{L_+}^+) \mid \partial\!\!\!/_C \eta = 0\right\}$.

Definition 3.2. Fix an integer $l > 2k$. The perturbed Cliffordian ASD-equation parameterized by $\eta \in H$ is defined by

$$\partial\!\!\!/_C \xi = -\sqrt{-1}\xi_\# \xi + \eta, \tag{2}$$

for $\xi \in L_l^2(C_{L_-}^-)$, where $\partial\!\!\!/_C \colon L_l^2(C_{L_-}^-) \longrightarrow L_l^2(C_{L_+}^+)$.

Definition 3.3. We define the space $\mathcal{S}_{\mathrm{ASD},\eta}$ by

$$\mathcal{S}_{\mathrm{ASD},\eta} = (D_1 + Q_1 - \eta)^{-1}(0).$$

In particular, we define

$$\mathcal{S}_{\mathrm{ASD}} = \mathcal{S}_{\mathrm{ASD},0} = (D_1 + Q_1)^{-1}(0).$$

Theorem 3.1. *If $\mathcal{S}_{ASD,\eta}$ exists, then $\mathcal{S}_{ASD,\eta}$ is a smooth oriented compact manifold of*

$$\dim \mathcal{S}_{ASD,\eta} = -\frac{\chi(X) + sign(X)}{2} + 1$$

for a generic perturbation $\eta \in H$ satisfying $\eta \neq 0$. Also $\mathcal{S}_{ASD}$ is compact.

Corollary 3.1. *$\mathcal{S}_{ASD,\eta} = \emptyset$ for a generic perturbation $\eta \in H$ satisfying $\eta \neq 0$.*

Proof. The virtual dimension of $\mathcal{S}_{\mathrm{ASD},\eta}$ is

$$-\frac{\chi(X)+sign(X)}{2}+1=-(b^0+b^2+\cdots+b^{2k}_+)+1<0,$$

by Assumption 3.1. □

The proof of Theorem 3.1 is given in Sections 4-5.

Definition 3.4. For $(\phi,\xi)\in\Gamma(S^+_L)\oplus\Gamma(C_{L^-}^-)$, the Cliffordian monopole equation is defined by

$$\begin{cases}\partial\!\!\!/_S\phi=\xi\phi,\\ \partial\!\!\!/_C\xi=-(\phi\otimes\phi^*)_0-\sqrt{-1}\xi_{\#}\xi\end{cases}, \tag{3}$$

where $(\phi\otimes\phi^*)_0$ is the purely imaginary part of $\phi\otimes\phi^*$, i.e.,

$$(\phi\otimes\phi^*)_0=\frac{1+\gamma}{2^{6k-2}}\sum_{\substack{I,J:\text{odd,type A},\\ e_Ie_J=-e_Je_I}}\langle e_Ie_J\phi,\phi\rangle e_Ie_J.$$

We have $\langle(\phi\otimes\phi^*)_0\phi,\phi\rangle=\frac{1}{2}|\phi|^4$. See Appendix 2.

Let $V=\Gamma(S^+_L)\oplus\Gamma(C_{L^-}^-)$ and $W=\Gamma(S^-_L)\oplus\Gamma(C_{L^+}^+)$. Let $D=\begin{pmatrix}\partial\!\!\!/_S & 0\\ 0 & \partial\!\!\!/_C\end{pmatrix}:V\longrightarrow W$. The quadratic map $Q\colon V\longrightarrow W$ is defined by

$$Q\begin{pmatrix}\phi\\ \xi\end{pmatrix}=\begin{pmatrix}-\xi\phi\\ (\phi\otimes\phi^*)_0+\sqrt{-1}\xi_{\#}\xi\end{pmatrix}.$$

Then we can write the Cliffordian monopole equation as

$$D+Q\colon V\longrightarrow W. \tag{4}$$

Definition 3.5. The perturbed Cliffordian monopole equation perturbed by $\eta\in H$ is defined by

$$\begin{cases}\partial\!\!\!/_S\phi=\xi\phi,\\ \partial\!\!\!/_C\xi=-(\phi\otimes\phi^*)_0-\sqrt{-1}\xi_{\#}\xi+\eta\end{cases} \tag{5}$$

for $(\phi,\xi)\in L^2_l(S^+_L)\oplus L^2_l(C_{L^-}^-)$.

Definition 3.6. We define the space $\mathcal{S}_{\mathrm{MON},\eta}$ of all solutions of the perturbed Cliffordian monopole equation perturbed by η by

$$\mathcal{S}_{\mathrm{MON},\eta}=(D+Q-\eta)^{-1}(0).$$

Furthermore we define the irreducible part ${\mathcal{S}_{\mathrm{MON},\eta}}^*$ of $\mathcal{S}_{\mathrm{MON},\eta}$ by

$${\mathcal{S}_{\mathrm{MON},\eta}}^* = \mathcal{S}_{\mathrm{MON},\eta} - \mathcal{S}_{\mathrm{ASD},\eta},$$

where the closed embedding

$$\mathcal{S}_{\mathrm{ASD},\eta} \hookrightarrow \mathcal{S}_{\mathrm{MON},\eta}$$

is defined by sending each $\xi \in \mathcal{S}_{\mathrm{ASD},\eta}$ to $(0,\xi) \in \mathcal{S}_{\mathrm{MON},\eta}$. In particular, we set $\mathcal{S}_{\mathrm{MON}} = \mathcal{S}_{\mathrm{MON},0}$.

Theorem 3.2. *If ${\mathcal{S}_{MON,\eta}}^*$ exists, then ${\mathcal{S}_{MON,\eta}}^*$ is a smooth oriented manifold of dimension*

$$\dim {\mathcal{S}_{MON,\eta}}^* = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X) + 1,$$

for a generic perturbation $\eta \in \Gamma({C_L}^+_+)$ satisfying $\eta \neq 0$ Furthermore, $\mathcal{S}_{MON,\eta}$ is compact. In particular, $\mathcal{S}_{MON}$ is compact. Here

$$ch(L) = \mathrm{tr}\left(\exp(-F_A)\right)$$

and

$$\hat{\mathcal{A}}(X) = \det{}^{\frac{1}{2}}\left(\frac{R/2}{\sinh(R/2)}\right),$$

where F_A is the curvature form of A.

Corollary 3.2. *${\mathcal{S}_{MON,\eta}}^* = \mathcal{S}_{MON,\eta}$ is compact for a generic perturbation $\eta \neq 0$.*

Proof. This lemma immediately follows from Corollary 3.1. □.

This theorem is proved in Sections 4-5.

The group $\mathrm{U}(1) = S^1 \subset \mathbb{C}$ acts on $\Gamma(S_L^\pm)$ by

$$\begin{array}{ccc} \Gamma(S_L^\pm) \times S^1 & \longrightarrow & \Gamma(S_L^\pm) \\ \phi \times e^{\sqrt{-1}\theta} & \longmapsto & e^{-\sqrt{-1}\theta}\phi \end{array}.$$

On the other hand S^1 acts trivially on $\Gamma({C_L}^\pm_\pm)$. When the line bundle L is trivial and $n = 8k+4, \quad k = 0,1,\ldots$, $\Gamma(S^\pm)$ is considered as an infinite dimensional $\mathbb{H}$-module such that $\Gamma(S^\pm) \cong \mathbb{H}^\infty$. Then the quaternion $j \in \mathbb{H}, j^2 = -1$ acts on $\Gamma(S_L^\pm)$ on the right via the multiplication Furthermore, j acts on $\Gamma({C_L}^\pm_\pm)$ by multiplying the number -1. Thus, in this case, we can consider the Lie group $\mathrm{Pin}(2) = \langle S^1, j\rangle \subset \mathbb{H}^\times$ acts on V and W on the right. It is easy to check that the (perturbed) Cliffordian monopole equation is S^1-equivariant. But, even if the line bundle L is trivial and $n = 8k+4, \quad k = 0,1,\ldots$, it is not Pin(2)-equivariant, because the term $-\sqrt{-1}\xi_\#\xi$ breaks the $\mathbb{Z}_2$-symmetry.

Definition 3.7. The moduli space $\mathcal{M}_\eta$ of $\mathcal{S}_{\mathrm{MON},\eta}$ is defined by

$$\mathcal{M}_\eta = \mathcal{S}_{\mathrm{MON},\eta}/S^1.$$

Since the circle group S^1 acts freely on $\mathcal{S}_{\mathrm{MON},\eta}$, $\mathcal{S}_{\mathrm{MON},\eta}$ is a principal S^1-bundle over $\mathcal{M}_\eta$, and $\mathcal{M}_\eta$ is a smooth oriented closed manifold of dimension

$$\dim \mathcal{M}_\eta = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X),$$

if $\mathcal{S}_{\mathrm{MON},\eta}$ exists. We remark that if $\mathcal{S}_{\mathrm{MON},\eta}$ exists, then dim $\mathcal{S}_{\mathrm{MON},\eta}$ must be greater than or equal to 1.

4. The manifold structure on $\mathcal{S}_{\mathrm{MON},\eta}$

Lemma 4.1. *There exists a Baire subset $\mathcal{B}$ of H such that for each $\eta \in \mathcal{B}$ satisfying $\eta \neq 0$, $\mathcal{S}_{ASD,\eta}$ is a smooth oriented manifold of dimension*

$$\dim \mathcal{S}_{ASD,\eta} = -\frac{\chi(X) + sign(X)}{2} + 1,$$

if $\mathcal{S}_{ASD,\eta}$ exists.

The proof of this lemma is the same as the proofs of the following two lemmas. We omit the proof of this lemma.

Lemma 4.2. *There exists a Baire subset $\mathcal{B}$ of H such that for each $\eta \in \mathcal{B}$ satisfying $\eta \neq 0$, ${\mathcal{S}_{MON,\eta}}^*$ is a smooth manifold of*

$$\dim {\mathcal{S}_{MON,\eta}}^* = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X) + 1,$$

if ${\mathcal{S}_{MON,\eta}}^*$ exists.

Proof. We define a map

$$J\colon L^2_l(S^+_L) \oplus L^2_l({C_L}^-) \oplus H \longrightarrow L^2_{l-1}(S^-_L) \oplus L^2_{l-1}({C_L}^+)$$

by

$$J(\phi, \xi, \eta) = (D + Q - \eta)\,(\phi, \xi).$$

Let $\mathcal{S} = J^{-1}(0)$. We denote by δJ the Fréchet differential of J at (ϕ, ξ, η). For smooth curves $\{\phi_t\} \subset L^2_l(S^+_L)$, $\{\xi_t\} \subset L^2_l({C_L}^-)$ and $\{\eta_t\} \subset H$ such that $\phi = \phi_0$, $\xi = \xi_0$ and $\eta = \eta_0$, we set

$$\dot{\phi} = \frac{d}{dt}\bigg|_{t=0} \phi_t, \quad \dot{\xi} = \frac{d}{dt}\bigg|_{t=0} \xi_t, \quad \dot{\eta} = \frac{d}{dt}\bigg|_{t=0} \eta_t,$$

respectively. The Fréchet differential δJ is given by

$$\delta J((\dot\phi,\dot\xi),\dot\eta) = \frac{d}{dt}\bigg|_{t=0} (\mathcal{D}+\mathcal{Q}-\eta_t)(\phi_t,\xi_t)$$
$$= \begin{pmatrix} \partial\!\!\!/_S\dot\phi - \dot\xi\phi - \xi\dot\phi \\ \partial\!\!\!/_C\dot\xi + (\dot\phi\otimes\phi^*)_0 + (\phi\otimes\dot\phi^*)_0 + \sqrt{-1}\dot\xi_\#\xi + \sqrt{-1}\xi\dot\xi_\# - \dot\eta \end{pmatrix}.$$

We have $\delta J = \delta_1 J \oplus \delta_2 J$, where

$$\delta_1 J(\dot\phi,\dot\xi) = \begin{pmatrix} \partial\!\!\!/_S\dot\phi - \dot\xi\phi - \xi\dot\phi \\ \partial\!\!\!/_C\dot\xi + (\dot\phi\otimes\phi^*)_0 + (\phi\otimes\dot\phi^*)_0 + \sqrt{-1}\dot\xi_\#\xi + \sqrt{-1}\xi\dot\xi_\# \end{pmatrix}$$

and

$$\delta_2 J(\dot\eta) = \begin{pmatrix} 0 \\ -\dot\eta \end{pmatrix}.$$

Since $\delta_1 J$ is an (real) elliptic operator, $\delta_1 J$ is Fredholm with

$$\mathrm{index}_{\mathbb{R}}\,\delta_1 J = 2\,\mathrm{index}_{\mathbb{C}}\left(\partial\!\!\!/_S\colon L^2_l(S^+_L)\longrightarrow L^2_{l-1}(S^-_L)\right)+\mathrm{index}_{\mathbb{R}}\left(\partial\!\!\!/_C\colon L^2_l(C_{L^-_-})\longrightarrow L^2_{l-1}(C_{L^+_+})\right).$$

By using Lemma 2.9 and the Atiyah-Singer index theorem, we have

$$\mathrm{index}_{\mathbb{R}}\delta_1 J = -\frac{\chi(X)+sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}}\int_X ch(L)\hat{\mathcal{A}}(X).$$

Let $\mathcal{R} = \frac{1+\gamma_X}{2}\sqrt{-1}\mathbb{R} \subset L^2_{l-1}(C_{L^+_+})$ and $\mathcal{L} = \mathcal{R}^\perp \subset L^2_{l-1}(C_{L^+_+})$. Now we show that the image of $\delta_1 J$ is contained in $L^2_{l-1}(S_L{}^-)\oplus\mathcal{L}$, and that $\delta_1 J\colon L^2_l(S_L{}^+)\oplus L^2_l(C_{L^-_-})\longrightarrow L^2_l(S_L{}^-)\oplus\mathcal{L}$ is surjective . We consider the formal adjoint operator $(\delta_1 J)^*$ of $\delta_1 J$. The restriction of $(\delta_1 J)^*$ on $0\oplus\mathcal{R}$ givn by

$$\begin{array}{ccc} 0\oplus\mathcal{R} & \longrightarrow & L^2_l(S_L{}^-)\oplus L^2_l(C_{L^-_-}) \\ (0,s) & \longmapsto & (s\phi,0) \end{array}.$$

Since $\phi\neq 0$, we have $\mathrm{Ker}\,(\delta_1 J)^*|_{\mathcal{R}} = 0$. Therefore we conclude $\mathcal{R}\subset\mathrm{Coker}\,\delta_1 J$. We assume that $\eta\neq 0$ then the perturbed Cliffordian monopole equation has a no trivial solution. Suppose that $\psi\in L_{l-1}(S^-_L)$ and $\nu\in\mathcal{L}$ are L^2-orthogonal to the images $\partial\!\!\!/_S\colon L^2_l(S^+_L)\longrightarrow L^2_{l-1}(S^-_L)$ and $\partial\!\!\!/_C\colon L^2_l(C_{L^-_-})\longrightarrow L^2_{l-1}(C_{L^+_+})$ respectively. Furthermore, suppose that (ψ,ν) is orthogonal to the image of $\delta_1 J$ and $(\psi,\nu)\neq 0$. By elliptic regularity, (ψ,ν) does not vanish on any open subset of X. Similarly (ϕ,ξ) also does not vanish on any open subset of X. Let $U\subset X$ be sufficiently small ball centered at a point x_0 where (ϕ,ξ) and (ψ,ν) are non-zero. We may assume that they are almost constant over U. Then there exists a vector $(\dot\phi,\dot\xi)\in(S^+_L\oplus C_{L^-_-})_{x_0}$ such that

$$\mathrm{Re}\left(\langle -\dot\xi\phi(x_0) - \xi(x_0)\dot\phi, \psi(x_0)\rangle\right)\neq 0$$

and

$$\mathrm{Re}\left(\langle(\dot\phi\otimes\phi^*(x_0))_0 + (\phi(x_0)\otimes\dot\phi^*(x_0))_0 + \sqrt{-1}(\dot\xi_\#\xi(x_0)+\xi(x_0)\dot\xi_\#),\nu(x_0)\rangle\right)\neq 0.$$

We can extend $(\dot{\phi}, \dot{\xi})$ to a global section $(\dot{\phi}, \dot{\xi}) \in \Gamma(S_L^+) \oplus \Gamma(C_{L^-}^-)$ vanishing outside U such that

$$\int_X \mathrm{Re}\left(\langle -\dot{\xi}\phi - \xi\dot{\phi}, \psi\rangle\right) dvol \neq 0$$

and

$$\int_X \mathrm{Re}\left(\langle (\dot{\phi} \otimes \phi^*)_0 + (\phi \otimes \dot{\phi}^*)_0 + \sqrt{-1}(\dot{\xi}_\# \xi + \xi \dot{\xi}_\#), \nu\rangle\right) dvol \neq 0,$$

by using a cut-off function. This means that (ψ, ν) is not orthogonal to $\delta_1 J(\dot{\phi}, \dot{\xi})$. This is contradiction. Thus $\delta_1 J$ is surjective. Now we apply the inverse function theorem for Banach spaces to our case. We conclude $\mathcal{S}$ is a Banach manifold. Here, we consider the following diagram:

$$\begin{array}{ccccc} \mathcal{S} & \hookrightarrow & L_l^2(S_L^+) \oplus L_l^2(C_{L^-}^-) \times H & \xrightarrow{J} & L_{l-1}^2(S_L^-) \oplus \mathcal{L} \\ \downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} & & , \\ H & = & H & & \end{array}$$

where π is the projection to the third factor and $\overline{\pi} = \pi|_{\mathcal{S}}$. Then we can write $\mathcal{S}_{\mathrm{MON},\eta} = \overline{\pi}^{-1}(\eta)$. Since $\mathcal{S}_{\mathrm{ASD},\eta} = \emptyset$, we have ${\mathcal{S}_{\mathrm{MON},\eta}}^* = \mathcal{S}_{\mathrm{MON},\eta}$. We consider the Fréchet differential $\delta\overline{\pi}$ of $\overline{\pi}$. Then

$$\mathrm{Ker}(\delta\overline{\pi}) = \left\{((\dot{\phi}, \dot{\xi}), \dot{\eta}) \mid \delta J((\dot{\phi}, \dot{\xi}), \dot{\eta}) = \dot{\eta} = 0\right\}$$

and

$$\begin{aligned} \mathrm{Im}(\delta\overline{\pi}) &= \left\{\dot{\eta} \mid \delta J((\dot{\phi}, \dot{\xi}), \dot{\eta}) = 0\right\} \\ &= (\delta_2 J)^{-1}(\mathrm{Im}(\delta_1 J)). \end{aligned}$$

Since $\delta_1 J$ is surjective, $\delta\overline{\pi}$ is surjective and

$$\mathrm{Ker}(\delta\overline{\pi}) = \mathrm{Ker}(\delta_1 J).$$

Therefore $\delta\overline{\pi}$ is a Fredholm map with

$$\begin{aligned} \mathrm{index}_{\mathbb{R}}\, \delta\overline{\pi} &= \mathrm{index}_{\mathbb{R}}\, \delta_1 J + \dim \mathcal{R} \\ &= -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X) + 1. \end{aligned}$$

Now we apply the infinite dimensional version of the Sard-Smale theorem to our case. There exists a Baire set $\mathcal{B} \subset H$ such that for each element $\eta \in \mathcal{B}$ with $\eta \neq 0$, the space ${\mathcal{S}_{\mathrm{MON},\eta}}^* = \overline{\pi}^{-1}(\eta)$ of the all solutions of the perturbed Cliffordian monopole equation is a smooth manifold of dimension

$$\dim {\mathcal{S}_{\mathrm{MON},\eta}}^* = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X) + 1.$$

□

Let $\mathcal{B}$ be the same as in Lemma 4.2

Lemma 4.3. *For each $\eta \in \mathcal{B}$ satisfying $\eta \neq 0$, ${\mathcal{S}_{MON,\eta}}^*$ is orientable. Furthermore, there is a one to one correspondence between the orientations of ${\mathcal{S}_{MON,\eta}}^*$ and those of*

$$\mathrm{H}^0_{DR}(X) \oplus \mathrm{H}^2_{DR}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{DR,+}(X)$$

as vector space over $\mathbb{R}$.

Proof. The tangent space of ${\mathcal{S}_{\mathrm{MON},\eta}}^*$ at a point $(\phi,\xi) \in {\mathcal{S}_{\mathrm{MON},\eta}}^*$ is isomorphic to the kernel of the elliptic operator

$$D_{(\phi,\xi)} = \begin{pmatrix} \partial\!\!\!/_S - \xi & - \bullet \phi \\ (\phi \otimes \bullet^*)_0 + (\bullet \otimes \phi^*)_0 & \partial\!\!\!/_C + \sqrt{-1}(\bullet_{\#}\xi + \xi_{\#}\bullet) \end{pmatrix}.$$

Let $\mathcal{D} = \left\{ D_{(\phi,\xi)} \mid (\phi,\xi) \in {\mathcal{S}_{\mathrm{MON},\eta}}^* \right\}$. Then $\mathcal{D}$ is an elliptic family parameterized by ${\mathcal{S}_{\mathrm{MON},\eta}}^*$. Thus ${\mathcal{S}_{\mathrm{MON},\eta}}^*$ is orientable if and only if the index bundle $\mathrm{index}\,\mathcal{D}$ of $\mathcal{D}$ is orientable. It is a well-known fact that $\mathrm{index}\,\mathcal{D}$ is orientable if and only if the determinant line bundle $\det(\mathrm{index}\,\mathcal{D})$ is trivial. Since we may assume that $\mathrm{Coker} D_{(\phi,\xi)}$ is trivial for $(\phi,\xi) \in {\mathcal{S}_{\mathrm{MON},\eta}}^*$, we have

$$\det(\mathrm{index}\,\mathcal{D}) = \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \Lambda^{\max}\mathrm{Ker}\, D_{(\phi,\xi)}.$$

On the other hand, we set

$$D_{(\phi,\xi),t} = \begin{pmatrix} \partial\!\!\!/_S - \xi & -(1-t)\bullet \phi \\ (1-t)(\phi \otimes \bullet^*)_0 + (1-t)(\bullet \otimes \phi^*)_0 & \partial\!\!\!/_C + (1-t)\sqrt{-1}(\bullet_{\#}\xi + \xi_{\#}\bullet) \end{pmatrix}.$$

and $\mathcal{D}_t = \left\{ D_{(\phi,\xi),t} \mid (\phi,\xi) \in {\mathcal{S}_{\mathrm{MON},\eta}}^* \right\}$ for $t \in I = [0,1]$. Then $\{\mathcal{D}_t\}_{t\in I}$ is an elliptic family parameterized by ${\mathcal{S}_{\mathrm{MON},\eta}}^* \times I$ and $\det(\mathrm{index}\{\mathcal{D}_t\}_{t\in I})$ is a real line bundle over $\mathcal{S}_{\mathrm{MON},\eta} \times I$. The elliptic family $\{\mathcal{D}_t\}_{t\in I}$ is the homotopy of $\mathcal{D}_0 = \mathcal{D}$ to $\mathcal{D}_1$. We now show that $\det(\mathrm{index}\{\mathcal{D}_t\}_{t\in I})$ is trivial. By definition, $\det(\mathrm{index}\{\mathcal{D}_t\}_{t\in I})|_{t=1} = \det(\mathrm{index}\,\mathcal{D}_1)$ over ${\mathcal{S}_{\mathrm{MON},\eta}}^* \times \{1\}$. We have

$$\begin{aligned} \det(\mathrm{index}\,\mathcal{D}_1) &\cong \det\left(\mathrm{index}(\partial\!\!\!/_S - \xi)\right) \otimes \det\left(\mathrm{index}(\partial\!\!\!/_C)\right) \\ &\cong \det\left(\mathrm{index}(\partial\!\!\!/_S - \xi)\right) \otimes \det\left(\mathrm{index}((d+d^*)\colon \Omega^-_- \longrightarrow \Omega^+_+)\right). \end{aligned}$$

Since $\{(\partial\!\!\!/_S - \xi)\colon L^2_l(S^+_L) \longrightarrow L^2_{l-1}(S^-_L)\}$ is a family of complex linear operators, $\det_{\mathbb{C}}\left(\mathrm{index}(\partial\!\!\!/_S - \xi)\right)$ is a trivial complex line bundle over ${\mathcal{S}_{\mathrm{MON},\eta}}^*$ and naturally oriented. Furthermore, $\det\left(\mathrm{index}((d+d^*)\colon \Omega^-_- \longrightarrow \Omega^+_+)\right)$ is clearly trivial and the orientation is determined by the orientation of the determinant line bundle of the trivial family of the extended AHS-complex

$$0 \longrightarrow \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^{2k-1} \xrightarrow{\ d^+\ } \Omega^{2k}_+ \longrightarrow 0.$$

Therefore $\det(\mathrm{index}\,\mathcal{D}_1)$ is a trivial line bundle and the orientation of $\det(\mathrm{index}\,\mathcal{D}_1)$ is determined by an orientation of

$$\mathrm{H}^0_{\mathrm{DR}}(X) \oplus \mathrm{H}^1_{\mathrm{DR}}(X)^* \oplus \cdots \oplus \mathrm{H}^{2k-1}_{\mathrm{DR}}(X)^* \oplus \mathrm{H}^{2k}_{\mathrm{DR},+}(X).$$

Since $I = [0,1]$ is contractible, the same conclusion for $\det(\operatorname{index}\mathcal{D}_0) = \det(\operatorname{index}\mathcal{D})$ follows. By using Assumption 3.1, we have the assertion of the lemma. □

5. The compactness of $\mathcal{S}_{\mathrm{MON},\eta}$

We will show the compactness of the space of all solutions of a perturbed Cliffordian monopole equation by using the standard elliptic bootstrap technique. The argument is the same as that in the proof of the compactness of the moduli space for the Seiberg-Witten equation. See Kronheimer-Mrowka [19], Morgan [25].

Since the chirality operator γ and the twisted Dirac operator $\partial\!\!\!/_C$ satisfy the relation $\partial\!\!\!/_C\gamma = -\gamma\partial\!\!\!/$, the following diagram is commutative.

Diagram 5.1. *The following diagram is commutative.*

$$\begin{array}{ccc} \Gamma(C_{L}{}_{-}^{-}) & \xrightarrow{\partial\!\!\!/_C} & \Gamma(C_{L}{}_{+}^{+}) \\ {\scriptstyle\frac{1-\gamma}{\sqrt{2}}}\uparrow & & \uparrow{\scriptstyle\frac{1+\gamma}{\sqrt{2}}}, \\ \Gamma(A_L) & \xrightarrow{\partial\!\!\!/_C} & \Gamma(C_L{}^{+}) \end{array}$$

where $A_L = A\otimes_{\mathbb{R}}AdP_L = \sqrt{-1}A$.

Lemma 5.1. *Suppose that* $(\phi,\xi) \in \Gamma(S_L^+) \oplus \Gamma(C_{L}{}_{-}^{-})$ *is a solution of the perturbed Cliffordian monopole equation (5). Then we have the following formulas*

$$\Delta_A\phi + c(F_A)\phi + \frac{\kappa}{4}\phi = -(\phi\otimes\phi^*)_0\phi + (1-\sqrt{-1})\xi_\#\xi\phi + \eta\phi \tag{6}$$

and

$$\Delta\xi + \frac{\kappa}{4}\xi = (\sqrt{-1}-1)(\xi(\phi\otimes\phi^*)_0 + (\phi\otimes\phi^*)_{0\#}\xi) - 2\xi\xi_\#\xi - \sqrt{-1}(\xi\eta + \eta_\#\xi), \tag{7}$$

where Δ_A *and* Δ *denote the Laplacians and* κ *the scalar curvature of* X.

Proof. In a local frame $(e_1,\dots,e_{4k})$ with respect to a normal chart U of X, i.e., $\nabla_i e_j = \nabla_{e_i}e_j = 0$ for all i,j, we write $\xi = \sum_{I:\text{odd}}\alpha^I e_I = \sum_{I:\text{type A}}\alpha^I(e_I - \gamma e_I) \in \Gamma(C_{L}{}_{-}^{-})$, where $\alpha^I \in L_l^2(X,\sqrt{-1}\mathbb{R})$ for each I (, so that $\langle\xi_\#\xi\phi\rangle = 2|\xi|^2|\phi|^2$). By Lemma 2.10, we have

$$\begin{aligned} \partial\!\!\!/_S{}^2\phi &= (\partial\!\!\!/_C\xi)\phi + \xi_\#\xi\phi \\ &= -(\phi\otimes\phi^*)_0\phi - \sqrt{-1}\xi_\#\xi\phi + \eta\phi + \xi_\#\xi\phi \\ &= -(\phi\otimes\phi^*)_0\phi + (1-\sqrt{-1})\xi_\#\xi\phi + \eta\phi. \end{aligned}$$

Therefore, by using the Lichnerowicz formula for the twisted Dirac operator $\partial\!\!\!/_S$, we have the first assertion of the lemma. Next, we have

$$\begin{aligned}\partial\!\!\!/_C{}^2\xi &= -\partial\!\!\!/_C(\phi\otimes\phi^*)_0 - \sqrt{-1}\partial\!\!\!/_C(\xi_\#\xi)\\ &= -((\partial\!\!\!/_C\phi)\otimes\phi^*)_0 - c((\phi\otimes{\nabla_A}^*\phi^*)_0) - \sqrt{-1}(\partial\!\!\!/_C\xi_\#)\xi - \sqrt{-1}c(\xi_\#\nabla\xi)\\ &= -((\partial\!\!\!/_C\phi)\otimes\phi^*)_0 - c((\phi\otimes{\nabla_A}^*\phi^*)_0) - \sqrt{-1}(\partial\!\!\!/_C\xi)_\#\xi - \sqrt{-1}c(\xi_\#\nabla\xi).\end{aligned}$$

We denote $-((\partial\!\!\!/_C\phi)\otimes\phi^*)_0$, $-c((\phi\otimes{\nabla_A}^*\phi^*)_0)$, $-\sqrt{-1}(\partial\!\!\!/_C\xi)_\#\xi$ and $-\sqrt{-1}c(\xi_\#\nabla\xi)$ by R_1, R_2, R_3 and R_4 respectively. We first compute R_1. We have

$$R_1 = -\xi(\phi\otimes\phi^*)_0,$$

by using the monopole equation (5). Let SD be a subset of the set of all multi-indices such that the set $\left\{\frac{e_I+\gamma e_I}{\sqrt{2}} \mid I\in SD\right\}$ forms an orthonormal basis of $C_{L_+^+}(U)$. Here we write $(\phi\otimes\nabla^*\phi)_0 = \sum_{i=1}^n\sum_{I\in SD} dx^i\beta_i^I\frac{e_I+\gamma e_I}{\sqrt{2}}$. Then we have

$$\begin{aligned}R_2 &= -\sum_{i=1}^n e_i\sum_{I\in SD}\beta_i^I\frac{e_I+\gamma e_I}{\sqrt{2}}\\ &= -\sum_{i=1}^n\sum_{I\in SD}\beta_i^I\frac{e_I+\gamma e_I}{\sqrt{2}}e_i + 2\sum_{i=1}^n\sum_{i\in I, I\in SD}\beta_i^I\frac{e_I-\gamma e_I}{\sqrt{2}}e_i\\ &= -\sum_{i=1}^n\sum_{I\in SD}\beta_i^I\frac{e_I+\gamma e_I}{\sqrt{2}}e_i + \sum_{i=1}^n\sum_{i\in I}\beta_i^I\frac{e_I-\gamma e_I}{\sqrt{2}}e_i,\end{aligned}$$

by the same technique as in the proof of Lemma 2.10. The first term of the right-hand side of the above equation is equal to zero, because it is in $\Gamma(C_{L_-^+})$ and the second term is $-(\phi\otimes(\partial\!\!\!/_S\phi)^*)_{0\#} = -(\phi\otimes\phi^*)_{0\#}\xi$. Moreover,

$$\begin{aligned}R_3 &= \sqrt{-1}(\phi\otimes\phi^*)_{0\#}\xi - (\xi_\#\xi)_\#\xi - \sqrt{-1}\eta_\#\xi\\ &= \sqrt{-1}(\phi\otimes\phi^*)_{0\#}\xi - \xi\xi_\#\xi - \sqrt{-1}\eta_\#\xi.\end{aligned}$$

Furthermore

$$\begin{aligned}
R_4 &= -\sqrt{-1}\sum_{i=1}^n e_i \sum_{I:\text{type A}} \alpha^I (e_I+\gamma e_I)\nabla_i\xi \\
&= \sqrt{-1}\left(\sum_{i=1}^n \sum_{I:\text{type A}} \alpha^I (e_I+\gamma e_I)e_i\nabla_i\xi - 2\sum_{i=1}^n \sum_{i\in I, I:\text{type A}} \alpha^I (e_I-\gamma e_I)e_i\nabla_i\xi\right) \\
&= \sqrt{-1}\left(\sum_{i=1}^n \sum_{I:\text{type A}} \alpha^I (e_I+\gamma e_I)e_i\nabla_i\xi - \sum_{i=1}^n \sum_{i\in I} \alpha^I (e_I-\gamma e_I)e_i\nabla_i\xi\right) \\
&= \sqrt{-1}\xi_\#(\partial\!\!\!/_C\xi) - \sqrt{-1}\xi(\partial\!\!\!/_C\xi) \\
&= -\sqrt{-1}\xi(\partial\!\!\!/_C\xi) \\
&= -\sqrt{-1}\xi\left(-(\phi\otimes\phi^*)_0 - \sqrt{-1}\xi_\#\xi + \eta\right) \\
&= \sqrt{-1}\xi(\phi\otimes\phi^*)_0 - \xi\xi_\#\xi - \sqrt{-1}\xi\eta.
\end{aligned}$$

Therefore we have the second assertion of the lemma. □

We simply denote $\oplus_{i=1}^k \Lambda^{4i-3}T^*X\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$ by Λ_L^A and $\oplus_{i=1}^k \Omega^{4i-3}(X)\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$ by Ω_L^A. Now we consider the differential operator (superconnection) $\nabla_\xi\colon \Gamma({S_L}^+) \longrightarrow \Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)$ corresponding to a Clifford section $\xi\in\Gamma({C_L}_-^-)$. Let $\xi' = \left(\frac{1-\gamma}{\sqrt{2}}\right)^{-1}\xi \in \Gamma(A_L)$. Since the map $\frac{1-\gamma}{\sqrt{2}}\colon \Gamma(A_L)\longrightarrow\Gamma({C_L}_-^-)$ is an isometric isomorphism, the section ξ' is uniquely determined by ξ. Let $\rho = c^{-1}\colon \Gamma(C_L)\longrightarrow \Omega^*(X)\otimes_{\mathbb{R}}\sqrt{-1}\mathbb{R}$ and $\alpha=\rho(\xi')$. Then $\alpha\in\Omega_L^A$. We define ∇_ξ by $\nabla_\xi = \nabla_A + \alpha$. We regard the space $\mathcal{A} = \nabla_A + \Omega_L^A$ as the affine space of $U(1)$-superconnections on ${S_L}^+$. The twisted Dirac operator is related to ∇_ξ by the relation $(\partial\!\!\!/_S+\xi)\phi = c(\nabla_\xi\phi)$. Let $\nabla_\xi^{\Lambda_L^A\otimes S_L^+}$ the differential operator $\nabla_A^{\Lambda_L^A\otimes S_L^+}+\alpha\colon \Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)\longrightarrow \Gamma(\Lambda_L^A\otimes_{\mathbb{R}}\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)$, where $\nabla_A^{\Lambda_L^A\otimes S_L^+}$ denote the covariant derivative on $\Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)$. The differential operator $\nabla_\xi^* = \nabla_A^* + \alpha^*\colon \Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)\longrightarrow\Gamma({S_L}^+)$ is defined by

$$\nabla_\xi^*\nabla_\xi\phi = -\mathrm{tr}\left(\nabla_\xi^{\Lambda_L^A\otimes S_L^+}\nabla_\xi\phi\right),$$

where $-\mathrm{tr}\left(\nabla_\xi^{\Lambda_L^A\otimes S_L^+}\nabla_\xi\phi\right)$ is the contraction of $\nabla_\xi^{\Lambda_L^A\otimes S_L^+}\nabla_\xi\phi$ with the Riemannian metric g. The Laplacian Δ_ξ is defined by

$$\Delta_\xi\phi = \nabla_\xi^*\nabla_\xi\phi$$

for all $\phi\in\Gamma({S_L}^+)$. More precisely,

$$\Delta_\xi\phi = \nabla_A^*\nabla_A\phi + \alpha^*\nabla_A\phi + \nabla^*(\alpha\phi) + \alpha^*\alpha\phi.$$

The operator $\alpha^*\colon \Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)\longrightarrow\Gamma({S_L}^+)$ is explicitly given by $\alpha^*\psi = -\mathrm{tr}(\alpha\psi)$ for all $\psi\in\Gamma(\Lambda_L^A\otimes_{\mathbb{R}}{S_L}^+)$. Thus we have $\alpha^*\alpha\phi = 2|\xi|^2\phi$.

Lemma 5.2. *The superconnection ∇_ξ is Hermitian, that is, the identity*

$$d|\phi|^2 = \langle \nabla_\xi \phi, \phi\rangle + \langle \phi, \nabla_\xi \phi\rangle$$

holds for all $\phi \in \Gamma({S_L}^+)$.

Proof. Write $\xi' = \sum_{|I|:\text{type A}} \alpha^I e_I$, where $\alpha^I \in C^\infty(X, \sqrt{-1}\mathbb{R})$. Then $\alpha = \sum_{|I|:\text{type A}} \alpha^I \rho(e_I)$. We have

$$\begin{aligned} d|\phi|^2 &= \langle \nabla_A \phi, \phi\rangle + \langle \phi, \nabla_A \phi\rangle \\ &= \langle \nabla_A \phi, \phi\rangle - \sum_{I:\text{type A}} \alpha^I \langle \rho(e_I)\phi, \phi\rangle + \sum_{I:\text{type A}} \alpha^I \langle \rho(e_I)\phi, \phi\rangle + \langle \phi, \nabla_A \phi\rangle \\ &= \langle \nabla_A \phi, \phi\rangle + \langle \alpha\phi, \phi\rangle + \langle \phi, \alpha\phi\rangle + \langle \phi, \nabla_A \phi\rangle \\ &= \langle (\nabla_A + \alpha)\phi, \phi\rangle + \langle \phi, (\nabla_A + \alpha)\phi\rangle \\ &= \langle \nabla_\xi \phi, \phi\rangle + \langle \phi, \nabla_\xi \phi\rangle. \end{aligned}$$

This completes the lemma. □

Lemma 5.3. (Kato's inequality) *For any $\phi \in L^2_l({S_L}^+)$ and $\xi \in L^2_l({C_L}^-_-)$, the inequality*

$$\Delta_X |\phi|^2 \leq 2\mathrm{Re}\langle \Delta_\xi \phi, \phi\rangle$$

holds almost everywhere on X, where Δ_X denotes the scalar Laplacian on X.

Proof. By using Lemma 5.2, we have

$$\Delta_X |\phi|^2 + 2|\nabla_\xi \phi|^2 = 2\mathrm{Re}\langle \Delta_\xi \phi, \phi\rangle.$$

Therefore we have the lemma. □

We simply denoted by $\partial\!\!\!/_\xi$ the twisted Dirac operator $\partial\!\!\!/_S + \xi$. We consider the total twisted Dirac operator

$$D\!\!\!/_S = \begin{pmatrix} 0 & \partial\!\!\!/_S^* \\ \partial\!\!\!/_S & 0 \end{pmatrix} : \begin{matrix} \Gamma({S_L}^+) \\ \oplus \\ \Gamma({S_L}^-) \end{matrix} \longrightarrow \begin{matrix} \Gamma({S_L}^+) \\ \oplus \\ \Gamma({S_L}^-) \end{matrix}.$$

Here the twisted Dirac operator $\partial\!\!\!/_S^* \colon \Gamma({S_L}^-) \longrightarrow \Gamma({S_L}^+)$ is the formal adjoint operator of $\partial\!\!\!/_S$, that is, $\langle \partial\!\!\!/_S \phi, \psi\rangle_{L^2} = \langle \phi, \partial\!\!\!/_S^* \psi\rangle_{L^2}$ holds for all pair $\phi \in \Gamma({S_L}^+)$ and $\psi \in \Gamma({S_L}^-)$. The formal adjoint $\partial\!\!\!/_\xi^* \colon \Gamma({S_L}^-) \longrightarrow \Gamma({S_L}^+)$ of $\partial\!\!\!/_\xi$ is given by $\partial\!\!\!/_\xi^* = \partial\!\!\!/_S + \xi_\#$, because the fact that

$$\begin{aligned} \langle \partial\!\!\!/_\xi \phi, \psi\rangle &= \langle \partial\!\!\!/_S \phi + \xi\phi, \psi\rangle \\ &= \langle \partial\!\!\!/_S \phi, \psi\rangle + \langle \xi\phi, \psi\rangle \\ &= \langle \phi, \partial\!\!\!/_S^* \psi\rangle + \langle \phi, \xi_\# \psi\rangle \\ &= \langle \phi, \partial\!\!\!/_S^* \psi + \xi_\# \psi\rangle \end{aligned}$$

holds. We have

$$\not{\mathcal{D}}_S{}^2 = \begin{pmatrix} \not{\partial}_S^*\not{\partial}_S & 0 \\ 0 & \not{\partial}_S\not{\partial}_S^* \end{pmatrix}.$$

We simply denote $\not{\partial}_S^*\not{\partial}_S \colon \Gamma({S_L}^+) \longrightarrow \Gamma({S_L}^+)$ by $\not{\partial}_S^2$. Similarly we denote $\not{\partial}_\xi^*\not{\partial}_\xi$ by $\not{\partial}_\xi^2$.

Definition 5.1. For any $(\phi, \xi) \in \Gamma({S_L}^+) \oplus \Gamma({C_L}_-^-)$, we define $\mathcal{F}(\phi, \xi) \in \Gamma({S_L}^+)$ by

$$\mathcal{F}(\phi, \xi) = \not{\partial}_\xi^2\phi - \left(\Delta_\xi\phi + c(F_A)\phi + \frac{\kappa}{4}\phi + \hat{\delta}(\xi)\phi\right).$$

Also we can write

$$\mathcal{F}(\phi, \xi) = \xi_\#(\not{\partial}_S\phi) + \not{\partial}_S^*(\xi\phi) - \alpha^*\nabla_A\phi - \nabla_A^*(\alpha\phi).$$

Lemma 5.4. *For any $(\phi, \xi) \in \Gamma({S_L}^+) \oplus \Gamma({C_L}_-^-)$ satisfying the equation $\not{\partial}_S\phi = \xi\phi$, we have the equality*

$$\Delta_{-\xi}\phi - \Delta_A\phi - \mathcal{F}(\phi, \xi) + (\not{\partial}_S\xi)\phi + 2|\xi|^2\phi + 2\hat{\delta}(\xi)\phi = 0.$$

Proof. Since $\not{\partial}_{-\xi}^2\phi = 0$, we have

$$\Delta_{-\xi}\phi + c(F_A)\phi + \frac{\kappa}{4} + \mathcal{F}(\phi, -\xi) + \hat{\delta}(-\xi)\phi = 0.$$

Also by Lemma 2.10 and the Lichnerowicz formula, we have

$$\Delta_A\phi + c(F_A)\phi + \frac{\kappa}{4} = (\not{\partial}_C\xi)\phi + \xi_\#\xi\phi.$$

From the above two equality, we have

$$\begin{aligned} 0 &= \Delta_{-\xi}\phi - \Delta_A\phi + \mathcal{F}(\phi, -\xi) + (\not{\partial}_C\xi)\phi + \xi_\#\xi\phi + \hat{\delta}(-\xi)\phi \\ &= \Delta_{-\xi}\phi - \Delta_A\phi - \mathcal{F}(\phi, \xi) + (\not{\partial}_C\xi)\phi + 2|\xi|^2\phi + 2\hat{\delta}(\xi)\phi, \end{aligned}$$

by $\mathcal{F}(\phi, -\xi) = -\mathcal{F}(\phi, \xi)$ and $\hat{\delta}(-\xi) = \hat{\delta}(\xi)$. □

Lemma 5.5. *For any $(\phi, \xi) \in \Gamma({S_L}^+) \oplus \Gamma({C_L}_-^-)$ satisfying the equation $\not{\partial}_S\phi = \xi\phi$, we have*

$$\mathcal{F}(\phi, \xi) = (\not{\partial}_C\xi)\phi - (\not{\partial}_C\xi)_{[0]}\phi + 2\hat{\delta}(\xi)\phi,$$

where we denote $(\bullet)_{[0]}$ by the $\Gamma(\frac{1+\gamma}{2}(C_L^0 \oplus C_L^{4k}))$ component of $\bullet \in \Gamma({C_L}_+^+)$.

Proof. Since $c(\nabla_A\phi) = \frac{1-\gamma}{2}c(\alpha)\phi$, we have

$$\begin{aligned}\alpha^*\nabla_A\phi &= -\mathrm{tr}(\alpha\nabla_A\phi)\\ &= (c(\alpha)c(\nabla_A\phi))_{[0]}\\ &= \left(\frac{1+\gamma}{2}c(\alpha)\frac{1-\gamma}{2}c(\alpha)\right)_{[0]}\phi\\ &= (\xi_{\#}\xi)_{[0]}\phi\\ &= 2|\xi|^2\phi.\end{aligned}$$

Furthermore we have

$$\begin{aligned}\nabla_A^*(\alpha\phi) &= -\mathrm{tr}\left(\nabla_A^{\Lambda_L^A\otimes S_L{}^+}(\alpha\phi)\right)\\ &= -\mathrm{tr}\left((\nabla_A^{\Lambda_L^A}\alpha)\phi + \alpha\nabla_A\phi\right)\\ &= (\partial\!\!\!/_C\xi)_{[0]}\phi + (\xi_{\#}\xi)_{[0]}\phi\\ &= (\partial\!\!\!/_C\xi)_{[0]}\phi + 2|\xi|^2\phi.\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathcal{F}(\phi,\xi) &= \xi_{\#}\partial\!\!\!/_S\phi + \partial\!\!\!/_S^*(\xi\phi) - \alpha^*\nabla_A\phi - \nabla_A^*(\alpha\phi)\\ &= \xi_{\#}\xi\phi + \partial\!\!\!/_S^2\phi - 2|\xi|^2\phi - (\partial\!\!\!/_C\xi)_{[0]}\phi - 2|\xi|^2\phi\\ &= 2\xi_{\#}\xi\phi + (\partial\!\!\!/_C\xi)\phi - 4|\xi|^2\phi - (\partial\!\!\!/_C\xi)_{[0]}\phi\\ &= (\partial\!\!\!/_C\xi)\phi - (\partial\!\!\!/_C\xi)_{[0]}\phi + 2\hat{\delta}(\xi)\phi.\end{aligned}$$

□

Lemma 5.6. *For any solution (ϕ,ξ) of the perturbed Cliffordian monopole equation (5), we have the equality*

$$\Delta_{-\xi}\phi - \Delta_A\phi + 2(1-\sqrt{-1})|\xi|^2\phi + \eta\phi = 0.$$

Proof. By lemmas 5.4 and 5.5, we have

$$\Delta_{-\xi}\phi - \Delta_A\phi + 2|\xi|^2\phi + (\partial\!\!\!/_C\xi)_{[0]}\phi.$$

Furthermore, by the second equation of (5), we have

$$(\partial\!\!\!/_C\xi)_{[0]}\phi = -2\sqrt{-1}|\xi|^2\phi + \eta\phi.$$

Therefore we have the assertion of the lemma. □

The following is the key lemma for the proof of the compactness.

Lemma 5.7. *For all* $(\phi, \xi) \in \mathcal{S}_{MON,\eta}$, *we have*

$$|\phi|^2 < C_1(X, L, g, h, A), \quad |\xi|^2 < C_2(X, L, g, h, A, \eta),$$

where C_1 *and* C_2 *are constants independent of the choice of* (ϕ, ξ) *and depending only on the Riemannian metric* g *of* X *, the Hermitian metric* h *on* L, *the Hermitian connection* A *and the perturbation* η.

Proof. By Lemma 5.1 and Lemma 5.6, we have

$$\Delta_{-\xi}\phi + c(F_A)\phi + \frac{\kappa}{4}\phi - (1-\sqrt{-1})\hat{\delta}(\xi)\phi + (\phi \otimes \phi^*)_0\phi = 0. \tag{8}$$

Since X is compact, there exists a point $x_0 \in X$ such that $|\phi(x)|$ takes the maximum value at x_0. By Kato's inequality, we have

$$\mathrm{Re}\langle \Delta_{-\xi}\phi(x_0), \phi(x_0)\rangle \geq 0.$$

Furthermore, by Lemma 2.12, we have

$$\langle -\hat{\delta}(\xi(x_0))\phi(x_0), \phi(x_0)\rangle \geq 0.$$

Therefore we have by (8)

$$\mathrm{Re}\langle c(F_A(x_0))\phi(x_0), \phi(x_0)\rangle + \frac{\kappa(x_0)}{4}|\phi(x_0)|^2 + \frac{1}{2}|\phi(x_0)|^4 \leq 0.$$

We define

$$||c(F_A)|| = \max_{x\in X}\left\{ \max_{||v||=1, v\in C_{L}{}^{-}_{-x}} \langle c(F_A)v, v\rangle \right\}.$$

and

$$\kappa_0 = \min_{x\in X}\{\kappa(x)\}.$$

Then we have $\phi \equiv 0$. Otherwise

$$\frac{1}{2}|\phi(x_0)|^2 \leq \max\left\{0, -\frac{\kappa_0}{4} + ||c(F_A)||\right\}.$$

Therefore there exists a constant C_1 depending only on κ and F_A such that

$$|\phi|^2 \leq C_1.$$

Furthermore, this implies that the operator norm of $(\phi\otimes\phi^*)_0$ and $(\phi\otimes\phi^*)_{0\#}$ are bounded by some constant K_1 depending on C_1 and n, that is,

$$\mathrm{Re}\langle \xi(\phi\otimes\phi^*)_0, \xi\rangle \leq K_1|\xi|^2, \quad \mathrm{Re}\langle (\phi\otimes\phi^*)_{0\#}\xi, \xi\rangle \leq K_1|\xi|^2.$$

By Lemma 5.1,(7), we have

$$\mathrm{Re}\langle \Delta\xi, \xi\rangle + \frac{\kappa}{4}|\xi|^2 + \mathrm{Re}\langle \xi(\phi\otimes\phi^*)_0 + (\phi\otimes\phi^*)_{0\#}\xi, \xi\rangle + 2\langle \xi\xi_{\#}\xi, \xi\rangle + \sqrt{-1}\langle \xi\eta + \eta_{\#}\xi, \xi\rangle = 0.$$

There exists a point $x_1 \in X$ where $|\xi(x)|$ takes the maximum value at x_1. Then, by Kato's inequality, we have $\mathrm{Re}\langle \Delta\xi(x_1), \xi(x_1)\rangle \geq 0$. Moreover, by Lemma 2.13, we have $2\langle \xi(x_1)\xi_{\#}(x_1)\xi(x_1), \xi(x_1)\rangle \geq 4|\xi(x_1)|^4$. Therefore we conclude that

$$\frac{\kappa(x_1)}{4}|\xi(x_1)|^2 + 2K_1|\xi(x_1)|^2 + 4|\xi(x_1)|^4 + \mathrm{Re}(\sqrt{-1}\langle \xi(x_1)\eta(x_1) + \eta_{\#}(x_1)\xi(x_1), \xi(x_1)\rangle) \leq 0.$$

Here we have the following estimation

$$|\mathrm{Re}(\sqrt{-1}\langle \xi(x_1)\eta(x_1) + \eta_{\#}(x_1)\xi(x_1), \xi(x_1)\rangle)| \leq 2|\eta(x_1)||\xi(x_1)|^2.$$

We denote by $\eta' \in \mathbb{R}$ the number $\max_{x\in X}\{|\eta(x)|\}$. We define

$$C_2 = \max\left\{0, -\frac{\kappa_0}{16} - \frac{K_1}{2} + \frac{1}{2}\eta'\right\}.$$

Then we have

$$|\xi|^2 \leq C_2.$$

This completes the lemma. □

By the following lemma, we complete the proof of Theorem 3.2.

Lemma 5.8. *For all $\eta \in \Gamma(C_{L^+}^+)$, $\mathcal{S}_{MON,\eta}$ is compact.*

Proof. By Lemma 5.7, we can assume that (ϕ, ξ) is L^q-bounded for $q > n$. Then $D(\phi, \xi) = -(Q-\eta)(\phi, \xi)$ is also L^q_1-bounded. Since D is an elliptic operator, (ϕ, ξ) is L^q_1-bounded. Since $(1-(n/q)) + (1-(n/q)) \geq 1-(n/q)$, we have $L^q_1 \times L^q_1 \subset L^q_1$ by the Sobolev multiplication theorem. Thus $(Q-\eta)(\phi, \xi)$ is also L^q_1-bounded. By induction, we conclude (ϕ, ξ) is L^q_{2k}-bounded. Applying the Sobolev multiplication theorem and the Sobolev embedding theorem to our case where $(2k-(n/q)) + (2k-(n/q)) \geq 2k-2k$, $n-(n/q) \geq 2k-2k$ and $2k \geq 2k$, we have $L^q_{2k} \times L^q_{2k} \subset L^2_{2k}$ and $L^q_{2k} \subset L^2_{2k}$. Thus we have both (ϕ, ξ) and $(Q-\eta)(\phi, \xi)$ are L^2_{2k}-bounded. The equation $D(\phi, \xi) = -(Q-\eta)(\phi, \xi)$ implies $D(\phi, \xi)$ is L^2_{2k}-bounded. The Sobolev multiplication theorem shows $L^2_{2k+1} \times L^2_{2k+1} \subset L^2_{2k+1}$, because $(2k+1-2k) + (2k+1-2k) \geq (2k+1-2k)$. Thus both $(Q-\eta)(\phi, \xi)$ and $D(\phi, \xi)$ are L^2_{2k+1}-bounded. By induction, (ϕ, ξ) is L^2_l-bounded for any $l > 2k$. Now we use the compactness of the embeddings $L^2_l \subset C^{l-2k-1}$. Therefore $\mathcal{S}_{\mathrm{MON},\eta}$ is compact. □

Remark 5. The embedding $\mathcal{S}_{\mathrm{ASD},\eta} \hookrightarrow \mathcal{S}_{\mathrm{MON},\eta}$ is closed for each $\eta \in \Gamma(C_{L^+}^+)$. Therefore $\mathcal{S}_{\mathrm{ASD},\eta}$ is compact. This completes the proof of Theorem 3.1.

Remark 6. By the proof of Lemma 5.8, we also conclude that if the line bundle L is trivial and the scalar curvature κ is positive then $\mathcal{S}_{\mathrm{MON},\eta}$ with sufficiently small perturbation η is empty.

Remark 7. The diffeomorphism class of $\mathcal{S}_{\mathrm{MON},\eta}$ is uniquely determined and is independent of the choice of $l > 2k$. Furthermore we can consider each $(\phi, \xi) \in \mathcal{S}_{\mathrm{MON},\eta}$ to be a C^∞-solution. Therefore $\mathcal{S}_{\mathrm{MON},\eta}$ is compact in $\Gamma(S_L{}^+) \oplus \Gamma(C_L{}_-^-)$ with respect to the C^∞-topology.

6. Smooth invariants of a spin manifold

In this section, we consider the perturbed moduli space $\mathcal{M}_\eta$ perturbed by a perturbation $\eta \in \mathcal{B} \cap \Gamma(C_L{}_+^+)$. In Section 4, we considered a family of linear elliptic operators

$$\left\{ \begin{pmatrix} \partial\!\!\!/_S - \xi & -\bullet\,\phi \\ (\bullet \otimes \phi^*)_0\xi + (\phi \otimes \bullet^*)_0\xi & \partial\!\!\!/_C + \sqrt{-1}\bullet_\#\xi + \sqrt{-1}\xi_\#\bullet \end{pmatrix} : \begin{matrix} L^2_l(S_L^+) \\ \oplus \\ L^2_l(C_L{}_-^-) \end{matrix} \longrightarrow \begin{matrix} L^2_{l-1}(S_L^-) \\ \oplus \\ L^2_{l-1}(C_L{}_+^+) \end{matrix} \right\}$$

parameterized by $(\phi, \xi) \in \mathcal{S}_{\mathrm{MON},\eta}$. Here we consider a simpler family of complex linear elliptic operators

$$\mathcal{D} = \left\{ \partial\!\!\!/_{(\phi,\xi)} : \Gamma(S_L) \longrightarrow \Gamma(S_L) \right\}$$

instead of the above family, where we define a full twisted Dirac operator $\partial\!\!\!/_{(\phi,\xi)}$ by

$$\partial\!\!\!/_{(\phi,\xi)} = \begin{pmatrix} 0 & (\partial\!\!\!/_S - \xi)^* \\ \partial\!\!\!/_S - \xi & 0 \end{pmatrix} : \begin{matrix} \Gamma(S_L^+) \\ \oplus \\ \Gamma(S_L^-) \end{matrix} \longrightarrow \begin{matrix} \Gamma(S_L^+) \\ \oplus \\ \Gamma(S_L^-) \end{matrix}.$$

Roughly speaking the index bundle $\operatorname{index} \mathcal{D}$ for the elliptic family $\mathcal{D}$ is formally given by

$$\begin{aligned} \operatorname{index} \mathcal{D} &= \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \mathrm{Ker}\,(\partial\!\!\!/_S - \xi) - \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \mathrm{Coker}\,(\partial\!\!\!/_S - \xi) \\ &= \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \mathrm{Ker}\,(\partial\!\!\!/_S - \xi) - \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \mathrm{Ker}\,(\partial\!\!\!/_S - \xi)^*. \end{aligned}$$

See Donaldson-Kronheimer [12],Section 5.2.1 and Atiyah-Singer [7]. The equation $\partial\!\!\!/_{(\phi,\xi)}\psi = 0$ for $\psi \in \Gamma(S_L)$ is S^1-invariant. Thus $\operatorname{index} \mathcal{D}$ is a virtual S^1-bundle over $\mathcal{S}_{\mathrm{MON},\eta}$, i.e., $[\operatorname{index} \mathcal{D}]_{S^1} \in K_{S^1}(\mathcal{S}_{\mathrm{MON},\eta})$. On the other hand, since S^1-acts freely on $\mathcal{S}_{\mathrm{MON},\eta}$, the quotient space $\operatorname{index} \mathcal{D}/S^1$ is a virtual bundle over $\mathcal{M}_\eta = \mathcal{S}_{\mathrm{MON},\eta}/S^1$, i.e., $[\operatorname{index} \mathcal{D}/S^1] \in K(\mathcal{M}_\eta)$. We simply denote $\operatorname{index} \mathcal{D}/S^1$ by $\operatorname{index} \hat{\mathcal{D}}$. The determinant line bundle $\det(\operatorname{index} \hat{\mathcal{D}})$ is given by

$$\det(\operatorname{index} \mathcal{D})/S^1 = \frac{\left\{ \bigcup_{(\phi,\xi)\in\mathcal{S}_{\mathrm{MON},\eta}} \Lambda^{\max}\mathrm{Ker}(\partial\!\!\!/_s - \xi) \otimes_{\mathbb{C}} (\Lambda^{\max}\mathrm{Coker}(\partial\!\!\!/_s - \xi))^* \right\}}{S^1}.$$

Then $\det(\operatorname{index} \hat{\mathcal{D}})$ is a complex line bundle over $\mathcal{M}_\eta$. Let $s \in \mathit{Spin}(X)$ be a given spin structure.

Definition 6.1. We define a rational number $q_s(X, L)$ by

$$q_s(X, L) = \int_{\mathcal{M}_\eta} ch(\text{index}\,\hat{\mathcal{D}}).$$

We also define an integer $q'_s(x, L)$ as follows: if $\dim \mathcal{M}_\eta$ is positive and even, then we set

$$q'_s(X, L) = \int_{\mathcal{M}_\eta} c_1(\det(\text{index}\,\hat{\mathcal{D}}))^d,$$

where $d = \frac{1}{2} \dim\, \mathcal{M}_\eta$; if $\dim \mathcal{M}_\eta = 0$, then we set

$$q'_s(X, L) = \int_{\mathcal{M}_\eta} 1 = \underset{\text{signed}}{\sharp}\, \mathcal{M}_\eta,$$

where $\underset{\text{signed}}{\sharp}\, \mathcal{M}_\eta$ means the signed number of points in $\mathcal{M}_\eta$, which are signed by orientation of $\mathcal{M}_\eta$; otherwise we set $q'_s(X, L) = 0$. Furthermore, we define two maps $q\colon Spin(X) \longrightarrow \mathbb{Q}$ and $q'\colon Spin(X) \longrightarrow \mathbb{Z}$ by $q(s) = q_s(X, L)$ and $q'(s) = q'_s(X, L)$ respectively.

We remark that

$$\int_{\mathcal{M}_\eta} ch(\text{index}\,\hat{\mathcal{D}}) = \langle ch([\text{index}\,\hat{\mathcal{D}}]), [\mathcal{M}_\eta]\rangle,$$

where the last "*ch*" means the Chern character homomorphism

$$ch\colon K(\mathcal{M}_\eta) \longrightarrow H^*(\mathcal{M}_\eta; \mathbb{Q}).$$

Theorem 6.1. *The rational number $q_s(X, L)$ is uniquely determined by the orientation preserving diffeomorphism type of X, the isomorphism class of L, the spin structure s and the choice of orientation of*

$$\mathrm{H}^0_{DR}(X) \oplus \mathrm{H}^2_{DR}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{DR,+}(X),$$

but independent of the choice of Riemannian metric g on X and the choice of Hermitian metric h on L.

Proof. Let g_0 and g_1 be two Riemannian metrics on X, and h_0 and h_1 two Hermitian metrics on L. Then there exists a smooth path $\{g_t\}$, $\ t \in I = [0, 1]$ of Riemannian metrics on X such that $\{g_t\}$ joins g_0 to g_1. Moreover there exists a smooth path $\{h_t\}$ of Hermitian metrics on L such that $\{h_t\}$ joins h_0 to h_1. Then we can choose a smooth path $\{A_t\}$ of connections on L such that for each $t \in I$, A_t is a Hermitian connection with respect to h_t. We denote by S^t_L and C^t_L the twisted spinor bundle and the Clifford bundle with respect to g_t and h_t respectively. Since $S^t_L \cong S_L$ and $C^t_L \cong C_L$ for any metrics $g_t, h_t \ \ (t \in I)$, we may identify all of them respectively. We denote by $\partial\!\!\!/^t_S\colon \Gamma(S_L) \longrightarrow \Gamma(S_L)$ and $\partial\!\!\!/^t_C\colon \Gamma(C_L) \longrightarrow \Gamma(C_L)$ the twisted Dirac operators with respect to the metrics g_t and h_t. And we denote by $Q_t\colon V \longrightarrow W$ the quadratic map with respect to the metrics

g_t and h_t. For each $t \in I$, we denote by γ_t the chirality operator with respect to g_t. We denote by $\mathrm{Eq}_t(\eta)$ the perturbed Cliffordian monopole equation perturbed by η with respect to the twisted Dirac operator $\partial\!\!\!/^t_S, \partial\!\!\!/^t_C$ and the quadratic map Q_t. For each $t \in I$, we denote by $\mathcal{B}_t$ the Baire subset of $H_t = \left\{\eta \in \Gamma(\frac{1+\gamma_t}{2} {C_L}^+) \mid \partial\!\!\!/^t_C \eta = 0\right\}$ such that for each $\eta \in \mathcal{B}_t$, the space of all solution of $\mathrm{Eq}_t(\eta)$ is a smooth manifold. Let $\eta_0 \in \mathcal{B}_0$ and $\eta_1 \in \mathcal{B}_1$.

Let

$$P = \left\{\mu \in L^2_l(I, {C_L}^+) \mid \mu(0) = \eta_0, \mu(1) = \eta_1, \mu(t) \in H_t \quad (t \in I)\right\}.$$

We define a map

$$J\colon L^2_l(S^+_L) \oplus L^2_l({C_L}^-_-) \oplus I \oplus P \longrightarrow L^2_{l-1}(S^-_L) \oplus L^2_{l-1}({C_L}^+_+) \oplus I$$

by

$$J(\phi, \xi, t, \mu) = ((D_t + Q_t - \eta_t)(\phi, \xi), t),$$

Let $\mathcal{S} = J^{-1}(\{0\} \oplus I)$. We can show that $\mathcal{S}$ is a Banach manifold , by the same method as in the proof of Lemma 4.1. Let $\pi\colon \mathcal{S} \longrightarrow P$ be the projection to the P factor. Again, by Assumption 3.1, we can show that there exists a Baire subset $\mathcal{C}$ of P such that for each $\mu \in \mathcal{C}$, the space $\hat{\mathcal{S}}_\mu = \pi^{-1}(\mu)$ is an oriented smooth compact manifold of dimension

$$\dim \hat{\mathcal{S}}_\mu = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X ch(L)\hat{\mathcal{A}}(X) + 2$$

with boundary $\mathcal{S}_{MON,\eta_1} - \mathcal{S}_{MON,\eta_0}$, by the same method as in the proof of Lemma 4.2. We choose one element $\mu \in \mathcal{C}$. We simply denote $\hat{\mathcal{S}}_\mu$ by $\hat{\mathcal{S}}$. Let $\pi'\colon \hat{\mathcal{S}} \longrightarrow I$ be the projection to the I factor. Let $\hat{\mathcal{M}} = \hat{\mathcal{S}}/S^1$. We have

$$\hat{\mathcal{M}} \subset \frac{L^2_l(S^+_+) \oplus L^2_l({C_L}^-_-)}{S^1} \oplus P \oplus I.$$

$\hat{\mathcal{M}}$ is oriented by the orientation of $X$and the choice of orientation of

$$\mathrm{H}^0_{\mathrm{DR}}(X) \oplus \mathrm{H}^2_{\mathrm{DR}}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{\mathrm{DR},+}(X)$$

and the natural orientation of I and L, by using the proof of Lemma 4.3. Then $\hat{\mathcal{M}}$ is a smooth compact manifold of dimension $\dim \hat{\mathcal{M}} = \dim \hat{\mathcal{S}} - 1$ with boundary $\mathcal{M}_{\eta_1} - \mathcal{M}_{\eta_0}$. Therefore $\mathcal{M}_{\eta_0}$ and $\mathcal{M}_{\eta_1}$ is oriented cobordant. Furthermore we have an elliptic family

$$\mathcal{D}_t = \left\{\partial\!\!\!/_{t,(\phi,\xi)} \mid (\phi, \xi) \in \mathcal{S}_t\right\},$$

where

$$\mathcal{S}_t = \hat{\mathcal{S}} \cap \left(L^2_l(S^+_L) \oplus L^2_l({C_L}^-_-) \times \{t\}\right)$$

and

$$\partial\!\!\!/_{t,(\phi,\xi)} = \begin{pmatrix} 0 & (\partial\!\!\!/^t_S - \xi)^* \\ \partial\!\!\!/^t_S - \xi & 0 \end{pmatrix} : \begin{matrix} \Gamma(S^+_L) \\ \oplus \\ \Gamma(S^-_L) \end{matrix} \longrightarrow \begin{matrix} \Gamma(S^+_L) \\ \oplus \\ \Gamma(S^-_L) \end{matrix}.$$

Then $\{\mathcal{D}_t\}$ is a homotopy of elliptic families which joins $\mathcal{D}_0$ to $\mathcal{D}_1$. Now we consider the index bundle

$$\mathrm{index}\, \hat{\mathcal{D}}_t = \frac{\mathrm{index}\, \mathcal{D}_t}{S^1}.$$

Since the Chern character of the index bundle of any elliptic family is homotopy invariant, therefore we have

$$\int_{\mathcal{M}_{\eta_0}} ch(\operatorname{index}\mathcal{D}_0) = \int_{\mathcal{M}_{\eta_1}} ch(\operatorname{index}\mathcal{D}_1).$$

See Atiyah-Singer [7] . This completes the proof of the theorem. □

Corollary 6.1. *The integer $q'_s(X, L)$ is uniquely determined by the orientation preserving diffeomorphism type of X, the isomorphism class of L and the spin structure s and the choice of orientation of*

$$\mathrm{H}^0_{DR}(X) \oplus \mathrm{H}^2_{DR}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{DR,+}(X),$$

but independent of the choice of Riemannian metric g on X and the choice of Hermitian metric h on L.

Corollary 6.2. *The maps $q\colon \mathrm{Spin}(X) \longrightarrow \mathbb{Q}$ and $q'\colon \mathrm{Spin}(X) \longrightarrow \mathbb{Z}$ are invariants of spin structure preserving diffeomorphism of X and isomorphism of L.*

Definition 6.2. We define an integer $q''_s(X, L)$ as follows: if $\dim \mathcal{M}_\eta$ is positive and even, then we set

$$q''_s(X, L) = \int_{\mathcal{M}_\eta} c_1(\mathcal{S}_{\mathrm{MON},\eta})^d;$$

if $\dim \mathcal{M}_\eta = 0$, then we set

$$q''_s(X, L) = \int_{\mathcal{M}_\eta} 1 = \underset{\text{signed}}{\sharp} \mathcal{M}_\eta;$$

otherwise we set $q''_s(X, L) = 0$.

Remark 8. In Definition 6.2, we denote by $c_1(\mathcal{S}_{\mathrm{MON},\eta})$ the 1st Chern class $c_1(L')$ of the complex line bundle L' over $\mathcal{M}_\eta$ associated with $\mathcal{S}_{\mathrm{MON},\eta}$.

Corollary 6.3. *The integer $q''_s(X, L)$ is uniquely determined by the orientation preserving diffeomorphism type of X, the isomorphism class of L and the spin structure s and the choice of orientation of*

$$\mathrm{H}^0_{DR}(X) \oplus \mathrm{H}^2_{DR}(X) \oplus \cdots \oplus \mathrm{H}^{2k}_{DR,+}(X),$$

but independent of the choice of Riemannian metric g on X and the choice of Hermitian metric h on L.

Corollaries 6.1-6.3 follow from Theorem 6.1 immediately.

Appendix 1

Our theory also goes well by the Spin^c case. Here we explain the outline briefly. Let X be a smooth connected closed Spin^c $4k$-manifold satisfying Assumption 3.1. Namely, there exist a principal $\mathrm{Spin}^c(4k)$-bundle P_{Spin^c} over X and a principal $\mathrm{U}(1)$-bundle P_L over X and a double covering map ρ of P_{Spin^c} onto $F_X \times P_L$ such that the diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}^c} & \xrightarrow{\rho} & F_X \times P_L \\ \overline{\pi}\Big\downarrow & & \pi\Big\downarrow \\ X & = & X \end{array}$$

is commutative, where $\overline{\pi}$ and π are the projections. We denoted by c the isomorphism class of the pair $(P_{\mathrm{Spin}^c}, P_L)$. An oriented manifold X has a Spin^c structure if and only if the condition $w_2(X) \equiv c_1(P_L) \mod 2$ is satisfied. We can identify a Spin^c-structure c to a cohomology class $c_1(P_L) \in H^2(X;\mathbb{Z})$ satisfying the above condition. Let L be a complex line bundle associated with P_L, i.e., $P_L \times_{\mathrm{U}(1)} \mathbb{C} \cong L$. We denote by W_L the Spin^c spinor bundle associated with the Spin^c structure c. Then W_L is locally isomorphic to $S \otimes_{\mathbb{C}} L^{\frac{1}{2}}$. We remark that the spinor bundle S and the square root $L^{\frac{1}{2}}$ always exist locally, even if X is not spin. We define the twisted Clifford bundle C_L by $C_L = C \otimes Ad\, P_L = \sqrt{-1}C$. The Riemannian metric g on X and the Hermitian metric h on L induce the twisted Dirac operators $\partial\!\!\!/_W \colon \Gamma(W_L) \longrightarrow \Gamma(W_L)$ and $\partial\!\!\!/_C \colon \Gamma(C_L) \longrightarrow \Gamma(C_L)$ as in Section 2. Then the Cliffordian monopole equation on the Spin^c manifold (X, c) is given by

$$\begin{cases} \partial\!\!\!/_W \phi = \xi\phi, \\ \partial\!\!\!/_C \xi = -(\phi \otimes \phi^*)_0 - \sqrt{-1}\xi_{\#}\xi \end{cases},$$

for $(\phi, \xi) \in \Gamma(W_L^+) \oplus \Gamma(C_L{}_-^-)$ in the same way as the spin case. The same results corresponding to ones in the spin case still hold in the Spin^c case.

Appendix 2

Let $C = C(\mathbb{R}^{4k})$ be the Clifford algebra on $\mathbb{R}^{4k}$ and $M = M^+ \oplus M^-$ the spinor module of C. In this section, we will show the following two formulas:

$$(\phi \otimes \phi^*)_0 = \frac{1+\gamma}{2^{6k-2}} \sum_{\substack{I,J:\text{odd,type A},\\ e_I e_J = -e_J e_I}} \langle e_I e_J \phi, \phi \rangle e_I e_J$$

and

$$\langle (\phi \otimes \phi^*)_0 \phi, \phi \rangle = \frac{1}{2}|\phi|^4$$

for $\phi \in M^+$, where we denote by $(\phi \otimes \phi^*)_0$ the purely imaginary part of $\phi \otimes \phi^*$ in $C^+ \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma A.1.

$$(\phi \otimes \phi^*)_0 = \frac{1}{2^{2k}} \sum_{K:\text{even, type A}} \langle e_K \phi, \phi \rangle e_K.$$

Proof. The orthogonal decomposition of $\phi \times \phi^*$ in $C^+ \otimes_{\mathbb{R}} \mathbb{C}$ is given by the formula

$$\begin{aligned}\phi \otimes \phi^* &= \sum_{K:\text{even, type A}} \langle e_K, \phi \otimes \phi^* \rangle_C e_K + \sum_{L:\text{even, type B}} \langle e_L, \phi \otimes \phi^* \rangle_C e_L \\ &= \frac{1}{2^{2k}} \left\{ \sum_{K:\text{even, type A}} \langle e_K \phi, \phi \rangle e_K + \sum_{L:\text{even, type B}} \langle e_L \phi, \phi \rangle e_L \right\},\end{aligned}$$

where $\langle \bullet, \bullet \rangle_C$ and $\langle \bullet, \bullet \rangle$ are the the metrics on $C \otimes_{\mathbb{R}} \mathbb{C}$ and M^+ respectively. For any even multi-index K of type A, we have

$$\langle e_K \phi, \phi \rangle = \langle {e_K}^2 \phi, e_K \phi \rangle = -\langle \phi, e_K \phi \rangle = -\overline{\langle e_K \phi, \phi \rangle}.$$

It follows that $\langle e_K \phi, \phi \rangle$ is purely imaginary. By the same way, for any even multi-index L of type B, we have

$$\langle e_L \phi, \phi \rangle = \langle {e_L}^2 \phi, e_L \phi \rangle = \langle \phi, e_L \phi \rangle = \overline{\langle e_L \phi, \phi \rangle}.$$

It follows that $\langle e_L \phi, \phi \rangle$ is purely real. Therefore the lemma is completed. □

Corollary A.2.

$$(\phi \otimes \phi^*)_0 = \frac{1+\gamma}{2^{6k-2}} \sum_{\substack{I,J:\text{odd, type A} \\ e_I e_J = -e_J e_I}} \langle e_I e_J \phi, \phi \rangle e_I e_J.$$

Proof. we can easily show that

$$(1+\gamma) \sum_{I,J:\text{odd, type A}} \langle e_I e_J \phi, \phi \rangle e_I e_J = 2^{4k-2} \sum_{K:\text{even}} \langle e_K \phi, \phi \rangle e_K,$$

where the number 2^{4k-2} is the multiplicity of each $\langle e_K \phi, \phi \rangle e_K$ in the left sum. For any pair of odd multi-indices I and J of type A, the product $e_I e_J$ is type A if and only if the equality $e_I e_J = -e_J e_I$ follows, and the product $e_I e_J$ is type B if and only if the equality $e_I e_J = e_J e_I$ follows. Therefore, by Lemma A.1, we have

$$\begin{aligned}(\phi \otimes \phi^*)_0 &= \frac{1}{2^{2k}} \sum_{K:\text{even, type A}} \langle e_K \phi, \phi \rangle e_K \\ &= \frac{1+\gamma}{2^{6k-2}} \sum_{\substack{I,J:\text{odd, type A} \\ e_I e_J = -e_J e_I}} \langle e_I e_J \phi, \phi \rangle e_I e_J.\end{aligned}$$

□

By Lemma A.1, we have

$$\begin{aligned}|\phi|^2 &= \langle(\phi\otimes\phi^*)\phi,\phi\rangle \\ &= \frac{1}{2^{2k}}\left\{\sum_{I:\text{even, type A}} \overline{\langle e_I\phi,\phi\rangle}\langle e_I\phi,\phi\rangle + \sum_{J:\text{even, type B}} \overline{\langle e_J\phi,\phi\rangle}\langle e_J\phi,\phi\rangle\right\} \\ &= \frac{1}{2^{2k}}\left\{\sum_{I:\text{even, type A}} |\langle e_I\phi,\phi\rangle|^2 + \sum_{J:\text{even, type B}} |\langle e_J\phi,\phi\rangle|^2\right\}.\end{aligned}$$

To prove the second formula, it suffice to show the following formula:

Lemma A.3.

$$\sum_{I:\text{even, type A}} |\langle e_I\phi,\phi\rangle|^2 = \sum_{J:\text{even, type B}} |\langle e_J\phi,\phi\rangle|^2.$$

Proof. We fix the usage of notation of the multi-indices as follows: we denote by I or I' an even multi-index of type A and denote by J or J' an even multi-index of type B. It suffice to consider the case where $\phi \neq 0$. Since

$$|\phi|^2\phi = (\phi\otimes\phi^*)\phi = \frac{1}{2^{2k}}\left\{\sum_I \langle e_I\phi,\phi\rangle e_I\phi + \sum_J \langle e_J\phi,\phi\rangle e_J\phi\right\},$$

we can decompose :

$$\sum_I \langle e_I\phi,\phi\rangle e_I\phi = C_A\phi + \psi_A$$

and

$$\sum_J \langle e_J\phi,\phi\rangle e_J\phi = C_B\phi + \psi_B,$$

where C_A and C_B are non-negative real numbers satisfying the relation $C_A + C_B = 2^{2k}|\phi|^2$, and ψ_A and ψ_B are some vectors in M^+ satisfying the relation $\psi_A + \psi_B = 0$ and $\langle\psi_A,\phi\rangle = \langle\psi_B,\phi\rangle = 0$.

We first consider the case where $\psi_A = \psi_B = 0$. Let $\mathcal{A} = \sum_I \langle e_I\phi,\phi\rangle e_I\phi = C_A\phi$ and $\mathcal{B} = \sum_J \langle e_J\phi,\phi\rangle e_J\phi = C_B\phi$. We have

$$\begin{aligned}C_A\mathcal{A} &= \sum_I \langle e_I\phi,\phi\rangle e_I \sum_{I'} \langle e_{I'}\phi,\phi\rangle e_{I'}\phi \\ &= \sum_{I,I'} \langle e_I\phi,\phi\rangle\langle e_{I'}\phi,\phi\rangle e_I e_{I'}\phi \\ &= \sum_I |\langle e_I\phi,\phi\rangle|^2\phi + \sum_{I\neq I'} \langle e_I\phi,\phi\rangle\langle e_{I'}\phi,\phi\rangle e_I e_{I'}\phi.\end{aligned}$$

Then we have

$$\begin{aligned}C_A^2|\phi|^2 &= \langle C_A\mathcal{A}, \phi\rangle \\ &= \sum_I |\langle e_I\phi, \phi\rangle|^2|\phi|^2 + \sum_{I\neq I'}\langle e_I\phi, \phi\rangle\langle e_{I'}\phi, \phi\rangle\langle e_I e_{I'}\phi, \phi\rangle \\ &= C_A|\phi|^4 + \sum_{\substack{I\neq I' \\ e_I e_{I'}:\text{type B}}} \langle e_I\phi, \phi\rangle\langle e_{I'}\phi, \phi\rangle\langle e_I e_{I'}\phi, \phi\rangle.\end{aligned}$$

Therefore we have

$$C_A^2|\phi|^2 - C_A|\phi|^4 = \sum_{\substack{I\neq I' \\ e_I e_{I'}:\text{type B}}} \langle e_I\phi, \phi\rangle\langle e_{I'}\phi, \phi\rangle\langle e_I e_{I'}\phi, \phi\rangle.$$

Similarly, we obtain

$$C_B^2|\phi|^2 - C_B|\phi|^4 = \sum_{\substack{J\neq J' \\ e_J e_{J'}:\text{type B}}} \langle e_J\phi, \phi\rangle\langle e_{J'}\phi, \phi\rangle\langle e_J e_{J'}\phi, \phi\rangle.$$

Here, since

$$\begin{aligned}&\sum_{\substack{I\neq I' \\ e_I e_{I'}:\text{type B}}} \langle e_I\phi, \phi\rangle\langle e_{I'}\phi, \phi\rangle\langle e_I e_{I'}\phi, \phi\rangle \\ &= \sum_{\substack{I\neq I' \\ e_I e_{I'}:\text{type B}}} \langle \phi, e_I\phi\rangle\langle e_{I'}\phi, e_I\phi\rangle\langle e_{I'}\phi, \phi\rangle \\ &= \sum_{\substack{I\neq I' \\ e_M = e_I e_{I'}:\text{type B}}} \overline{{}^t\phi}(\phi\otimes(e_M\phi)^*)^2\phi \\ &= \sum_{\substack{J\neq J' \\ e_M = e_J e_{J'}:\text{type B}}} \overline{{}^t\phi}(\phi\otimes(e_M\phi)^*)^2\phi \\ &= \sum_{\substack{J\neq J' \\ e_J e_{J'}:\text{type B}}} \langle \phi, e_J\phi\rangle\langle e_{J'}\phi, e_J\phi\rangle\langle e_{J'}\phi, \phi\rangle \\ &= \sum_{\substack{J\neq J' \\ e_J e_{J'}:\text{type B}}} \langle e_J\phi, \phi\rangle\langle e_{J'}\phi, \phi\rangle\langle e_I e_{J'}\phi, \phi\rangle,\end{aligned}$$

it follows that

$$\begin{aligned}0 &= C_A^2|\phi|^2 - C_A|\phi|^4 - C_B|\phi|^2 + C_B|\phi|^4\\ &= (C_B - C_A)|\phi|^2(C_A + C_B - |\phi|^2)\\ &= (2^{2k} - 1)|\phi|^4(C_A - C_B).\end{aligned}$$

Therefore we conclude that $C_A = C_B$.

We next consider the case where $\psi_A \neq 0$. We denote by $\tau\colon M^+ \longrightarrow M^+$ the $\mathbb{C}$-linear map uniquely determined by the condition $\tau(\phi) = \psi_A$, $\tau(\psi_A) = \phi$ and $\tau|_{\langle\{\phi,\psi_A\}\rangle^\perp} = 1$. Then the restriction of τ on $\langle\{\phi, \psi_A\}\rangle$ is given by

$$\tau|_{\langle\{\phi,\psi_A\}\rangle} = \begin{pmatrix} 0 & \frac{|\phi|}{|\psi_A|} \\ \frac{|\psi_A|}{|\phi|} & 0 \end{pmatrix}.$$

The linear map τ satisfies the properties: $\tau^2 = 1$ and $\langle \tau s, {}^t\tau t\rangle = \langle s, t\rangle$ for all $s, t \in M^+$. By definition of τ, we have

$$\tau(\sum_I \langle e_I\phi, \phi\rangle e_I\phi) = \tau(C_A\phi + \psi_A) = C_A\psi_A + \phi$$

and

$$\tau(\sum_J \langle e_J\phi, \phi\rangle e_J\phi) = \tau(C_B\phi + \psi_B) = \tau(C_B\phi - \psi_A) = C_B\psi_A - \phi.$$

The other way, we have , by linearity of τ,

$$\tau(\sum_I \langle e_I\phi, \phi\rangle e_I\phi) = \sum_I \langle e_I\phi, \phi\rangle \tau(e_I\phi)$$

and

$$\tau(\sum_J \langle e_J\phi, \phi\rangle e_J\phi) = \sum_J \langle e_J\phi, \phi\rangle \tau(e_J\phi).$$

Then we have

$$|\phi|^2 = \langle C_A\psi + \phi, \phi\rangle = \sum_I \overline{\langle e_I\phi, \phi\rangle}\langle \tau(e_I\phi), \phi\rangle$$

and

$$-|\phi|^2 = \langle C_B\psi - \phi, \phi\rangle = \sum_J \overline{\langle e_J\phi, \phi\rangle}\langle \tau(e_J\phi), \phi\rangle.$$

Therefore, we have

$$\begin{aligned}0 &= \sum_I \overline{\langle e_I\phi, \phi\rangle}\langle \tau(e_I\phi), \phi\rangle + \sum_J \overline{\langle e_J\phi, \phi\rangle}\langle \tau(e_J\phi), \phi\rangle\\ &= \sum_I \overline{\langle e_I\phi, \phi\rangle}\langle e_I\phi, {}^t\tau\phi\rangle + \sum_J \overline{\langle e_J\phi, \phi\rangle}\langle e_J\phi, {}^t\tau\phi\rangle\\ &= \frac{|\phi|}{|\psi_A|}\left(\sum_I \overline{\langle e_I\phi, \phi\rangle}\langle e_I\phi, \psi_A\rangle - \sum_J \overline{\langle e_J\phi, \phi\rangle}\langle e_J\phi, \psi_B\rangle\right).\end{aligned}$$

Since $|\phi|/|\psi_A| \neq 0$, we have

$$\begin{aligned}0 &= \sum_I \overline{\langle e_I\phi,\phi\rangle}\langle e_I\phi, \sum_{I'}\langle e_{I'}\phi,\phi\rangle e_{I'}\phi - C_A\phi\rangle \\ &\quad - \sum_J \overline{\langle e_J\phi,\phi\rangle}\langle e_J\phi, \sum_{J'}\langle e_{J'}\phi,\phi\rangle e_{J'}\phi - C_B\phi\rangle \\ &= \sum_{I,I'} \overline{\langle e_I\phi,\phi\rangle}\langle e_{I'}\phi,\phi\rangle\langle e_I\phi, e_{I'}\phi\rangle - C_A\sum_I |\langle e_I\phi,\phi\rangle|^2 \\ &\quad - \sum_{J,J'} \overline{\langle e_J\phi,\phi\rangle}\langle e_{J'}\phi,\phi\rangle\langle e_J\phi, e_{J'}\phi\rangle + C_B\sum_J |\langle e_J\phi,\phi\rangle|^2.\end{aligned}$$

Here, since we can show the following equality as in the first case

$$\sum_{I\neq I'}\langle e_I\phi,\phi\rangle\langle e_{I'}\phi,\phi\rangle\langle e_I\phi, e_{I'}\phi\rangle = \sum_{J\neq J'}\langle e_J\phi,\phi\rangle\langle e_{J'}\phi,\phi\rangle\langle e_J\phi, e_{J'}\phi\rangle,$$

and since $\sum_I |\langle e_I\phi,\phi\rangle|^2 = C_A|\phi|^2$ and $\sum_J |\langle e_J\phi,\phi\rangle|^2 = C_B|\phi|^2$, we have

$$\begin{aligned}0 &= C_A|\phi|^4 - C_A^2|\phi|^2 - C_B|\phi|^4 + C_B|\phi|^2 \\ &= (C_B - C_A)|\phi|^2(C_A + C_B - |\phi|^2) \\ &= (2^{2k} - 1)(C_B - C_A)|\phi|^4.\end{aligned}$$

Therefore we conclude that $C_A = C_B$. The lemma is completed. □

Remark 9. We expect that $\psi_A = \psi_B = 0$ in the proof of Lemma A.3, so that $(\phi\otimes\phi^*)_0\phi = \frac{1}{2}|\phi|^2\phi$. But we can not yet prove it. So this is a conjecture except 4-dimensional cases.

Appendix3

In this section, we consider 4-dimensional cases. Let X be a simply-connected closed Spin^c 4-manifold. Let L be a Hermitian line bundle over X and A a Hermitian connection on L. The twisted Dirac operators $\partial\!\!\!/_W$ and $\partial\!\!\!/_C$ are assumed to be constructed by using A. Let us assume $w_2(X) \equiv c_1(L) \mod 2$. We think $c = c_1(L)$ of a Spin^c structure on X. The perturbed Cliffordian monopole equation for (X, L) is given by

$$(9)\qquad \begin{cases}\partial\!\!\!/_W\phi = \xi\phi, \\ \partial\!\!\!/_C\xi = -(\phi\otimes\phi^*)_0 - 2\sqrt{-1}(1+\gamma)|\xi|^2 + \eta_1,\end{cases}$$

for $(\phi,\xi) \in \Gamma({W_L}^+) \oplus \Gamma({C_L}^-)$, where the perturbation term $\eta_1 \in \Gamma(C_{L+}^{+})$ satisfies $\partial\!\!\!/_C\eta_1 = 0$. By the way, the perturbed Seiberg-Witten equation for (X, c) with the Coulomb gauge condition is given by

$$\begin{cases} \partial\!\!\!/_W \phi = -a\phi, \\ d^+ a = -F_A^+ + c^{-1}(\phi \otimes \phi^*)_0 + \omega, \\ d^* a = 0, \end{cases} \tag{10}$$

for $(\phi, a) \in \Gamma(W_L{}^+) \oplus \sqrt{-1}\Omega^1(X)$, where $\omega \in \sqrt{-1}\Omega^2_+(X)$ satisfies $d\omega = d^*\omega = 0$. Here we write $\xi = -\frac{1-\gamma}{\sqrt{2}} c(a) \in \Gamma(C_{L^-}^-)$ and $\eta_0 = -c(\omega) \in \Gamma(C_{L^+}^+)$. Then the equation (10) is equivalent to the following equation:

$$\begin{cases} \partial\!\!\!/_W \phi = \xi\phi, \\ \partial\!\!\!/_C \xi = c(F_A^+) - (\phi \otimes \phi^*)_0 + \eta_0, \end{cases} \tag{11}$$

for $(\phi, \xi) \in \Gamma(W_L{}^+) \oplus \Gamma(C_{L^-}^-)$. We define a homotopy connecting (9) and (11) by

$$\begin{cases} \partial\!\!\!/_W \phi = \xi\phi, \\ \partial\!\!\!/_C \xi = (1-t)c(F_A^+) - (\phi \otimes \phi^*)_0 - 2\sqrt{-1}t(1+\gamma)|\xi|^2 + \eta(t), \end{cases} \tag{12}$$

for $t \in I = [0, 1]$ and $(\phi, \xi) \in \Gamma(W_L{}^+) \oplus \Gamma(C_{L^-}^-)$, where $\eta(t) \in \Gamma(C_{L^-}^-)$ is a smooth homotopy connecting η_0 and η_1 satisfying the condition $\partial\!\!\!/_C \eta(t) = 0$ for all $t \in I$. From the Seiberg-Witten theory, we can detect the generic path condition for the perturbation $\eta(t)$ of (12). If $b^2_+(X) \geq 2$, then we can show that the space

$$\hat{\mathcal{A}} = \left\{ (\phi, \xi, t) \in L^2_5(W^+) \oplus L^2_5(C_{L^-}^-) \oplus I \mid (\phi, \xi, t) \quad \text{satisfies (12)} \right\}$$

is a smooth oriented manifold with boundary of dimension

$$\dim \hat{\mathcal{S}} = -\frac{\chi(X) + sign(X)}{2} + \frac{2}{(2\pi\sqrt{-1})^{2k}} \int_X e^{\frac{c}{2}} \hat{\mathcal{A}}(X) + 1$$

for any generic perturbation $\eta(t)$. Furthermore we can show that the projection $p \colon \hat{\mathcal{S}} \longrightarrow I$ is surjection. Now we show the following proposition:

Proposition A.4. $\hat{\mathcal{S}}$ is compact.

Outline of the proof. For any solution (ϕ, ξ, t) of (12), we have the following equalities:

$$\Delta_A \phi + tc(F_A^+)\phi + \frac{\kappa}{4}\phi + (\phi \otimes \phi^*)_0 \phi = 0$$

and

$$\Delta\xi + \frac{\kappa}{4}\xi = (\sqrt{-1}t - 1)(\xi(\phi \otimes \phi^*)_0 + (\phi \otimes \phi^*)_{0\sharp}\xi) - 4t^2(1-\gamma)|\xi|^2\xi - \sqrt{-1}t(\xi\eta + \eta_\sharp\xi)$$

as the proof of Lemma 5.1. Using these equalities, as the proofs of Lemmas 5.4-5.7, we have the a priori bound estimations:

$$|\phi|^2 < C_1(X, L, g, h, A)$$

and

$$|\xi|^2 < C_2(X, L, g, h, A, \eta(t), t).$$

Therefore, for all $t \neq 0$, we can show that $\hat{\mathcal{S}}_t = p^{-1}(\{t\})$ is compact by using the elliptic bootstrap argument with respect to the equation (12) as the proof of Lemma 5.8. On the other hand, since the space $\hat{\mathcal{S}}_0$ is the space of all solutions of the Seiberg-Witten equation with the Coulomb gauge condition, we know the space $\hat{\mathcal{S}}_0$ is compact. Therefore $\hat{\mathcal{S}}$ is compact. $\square$

By Proposition A.4 , $\hat{\mathcal{S}}$ gives the oriented cobordism between $\hat{\mathcal{S}}_0$ and $\hat{\mathcal{S}}_1$. As a result of this, we obtain the following:

Theorem A.5. Let X be a simply-connected closed spin 4-manifold with $b_+^2(X) \geq 2$, and L a Hermitian line bundle over X with $c = c_1(L)$ which is equivalent to 0 modulo 2. Then our invariant $q_s''(X, L)$ is equal to the Seiberg-Witten invariant $SW(X, c)$.

Remark 10. The assertion of Theorem A.5 is also valid even if X is a simply-connected closed Spin^c-manifold with $b_+^2(X) \geq 2$. Therefore for any simply-connected closed smooth 4-manifold X with $b_+^2(X) \geq 2$, and for any Spin^c-structure c on X, our invariant $q''(X, c)$ is equal to the Seiberg-Witten invariant $SW(X, c)$.

Acknowledgments

The author is extremely grateful to Professor Mikio Furuta, for his kind advice. His lectures and his excellent paper [14] inspired the author with the main idea of this paper. Moreover, the author would like to thank Professor Masashi Ishida, Professor Yoshiyuki Koga and Doctor Seiji Nagami, for all their useful comments.

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