```
Maxima 5.39.0 http://maxima.sourceforge.net
using Lisp SBCL 1.3.13
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Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
```

```
(%i1) load(eigen);
```

(%o1) /usr/local/share/maxima/5.39.0/share/matrix/eigen.mac

```
(%i2) load("kihon.mac");
```

(%o2) kihon.mac

```
(%i5) /* Example 1 */
        A : matrix([-4, 3, -2], [2, 0, 0], [11, -6, 4]);
```

$$(\%o5)\quad \begin{pmatrix} -4 & 3 & -2 \\ 2 & 0 & 0 \\ 11 & -6 & 4 \end{pmatrix}$$

```
(%i6) A^^2;
```

$$(\%o6)\quad \begin{pmatrix} 0 & 0 & 0 \\ -8 & 6 & -4 \\ -12 & 9 & -6 \end{pmatrix}$$

```
(%i7) A^^3;
```

$$(\%o7)\quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i8) /* A is nilpotent, and has the only eigenvalue '0'  */
        cp_kaidan(A);
```

$$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 2 & 0 & 0 \\ 11 & -6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{3}{2} & -1 \\ 11 & -6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{3}{2} & -1 \\ 0 & \frac{9}{4} & -\frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{2}{3} \\ 0 & \frac{9}{4} & -\frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & \frac{9}{4} & -\frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

(%o9) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$

```
(%i10) rank(A);
```

(%o10) 2

```
(%i11) cp_kaidan(A^^2);
```

$\begin{pmatrix} -8 & 6 & -4 \\ 0 & 0 & 0 \\ -12 & 9 & -6 \end{pmatrix}$
$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 \\ -12 & 9 & -6 \end{pmatrix}$
$\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(%o11) $\begin{pmatrix} 1 & -\frac{3}{4} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

```
(%i12) rank(A^^2);
```

(%o12) 1

```
(%i13) /* W(0) = {v | A^k v = 0}, V(0) = {v | A v = 0}, V subsp. W,
          dim W = 3, dim V = 1, a0 in W \ V */
          a0 : matrix([0],[1],[0]);
```

(%o13) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

```
(%i14) /* a1 := A . a0 in W \ V */
          a1 : A . a0;
```

(%o14) $\begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}$

```
(%i15) /* a2 := A^^2 . a0 = A . a1 in V */
          a2 : A^^2 . a0;
```

(%o15) $\begin{pmatrix} 0 \\ 6 \\ 9 \end{pmatrix}$

```
(%i16) /* AA = (a2, a1, a0) is a basis of W = R^3 */
          AA : addcol(a2, a1, a0);
```

(%o16) $\begin{pmatrix} 0 & 3 & 0 \\ 6 & 0 & 1 \\ 9 & -6 & 0 \end{pmatrix}$

```
(%i17) /* as a matrix */
          Q : AA;
```

(%o18) $$\begin{pmatrix} 0 & 3 & 0 \\ 6 & 0 & 1 \\ 9 & -6 & 0 \end{pmatrix}$$

```
(%i19) P : myinv(Q);
```

$$\begin{pmatrix} 0 & 3 & 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 & 1 & 0 \\ 9 & -6 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 6 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & -6 & -\frac{3}{2} & 0 & -\frac{3}{2} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & -6 & -\frac{3}{2} & 0 & -\frac{3}{2} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 2 & -\frac{3}{2} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{9} & 0 & \frac{1}{9} \\ 0 & 1 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} & 1 & -\frac{2}{3} \end{pmatrix}$$

(%o19) $$\begin{pmatrix} \frac{2}{9} & 0 & \frac{1}{9} \\ \frac{1}{3} & 0 & 0 \\ -\frac{4}{3} & 1 & -\frac{2}{3} \end{pmatrix}$$

```
(%i20) /* The Jordan form of A */
          JA : P . A . Q;
```

(%o20) $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i21) kill(A, a0, a1, a2, Q, P, JA);
```

(%o21) **done**

```
(%i22) /* Example */
          A : matrix([2, 3, -2], [-3, 14, -7], [-5, 19, -9]);
```

(%o22) $$\begin{pmatrix} 2 & 3 & -2 \\ -3 & 14 & -7 \\ -5 & 19 & -9 \end{pmatrix}$$

```
(%i23) A^^3;
```

(%o23) $$\begin{pmatrix} 15 & 22 & -17 \\ -43 & 166 & -91 \\ -74 & 256 & -138 \end{pmatrix}$$

```
(%i24) /* A is not nilpotent. rho(x) : the eigen polynomial of A  */
       rho(x) := ratsimp(determinant(A - x * ident(3)));
```

(%o24) $\rho(x) := \mathrm{ratsimp}(\mathrm{determinant}(A - x\,\mathrm{ident}(3)))$

```
(%i25) rho(x);
```

(%o25) $-x^3 + 7\,x^2 - 16\,x + 12$

```
(%i26) factor(rho(x));
```

(%o26) $-(x-3)\,(x-2)^2$

```
(%i27) /* A has eigenvalues 3 (multiplicity 1) and 2 (multiplicity 2). */
       B : A - 2 * ident(3);
```

(%o27) $$\begin{pmatrix} 0 & 3 & -2 \\ -3 & 12 & -7 \\ -5 & 19 & -11 \end{pmatrix}$$

```
(%i28) rank(B);
```

(%o28) 2

```
(%i29) B2 : B^^2;
```

(%o29) $$\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -2 & 4 & -2 \end{pmatrix}$$

```
(%i30) rank(B2);
```

(%o30) 1

```
(%i31) C : A - 3 * ident(3);
```

(%o31) $$\begin{pmatrix} -1 & 3 & -2 \\ -3 & 11 & -7 \\ -5 & 19 & -12 \end{pmatrix}$$

```
(%i32) rank(C);
```

(%o32) 2

```
(%i33) cp_kaidan(B);
```

$$\begin{pmatrix} -3 & 12 & -7 \\ 0 & 3 & -2 \\ -5 & 19 & -11 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -4 & \frac{7}{3} \\ 0 & 3 & -2 \\ -5 & 19 & -11 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -4 & \frac{7}{3} \\ 0 & 3 & -2 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$$
$$\begin{pmatrix} 1 & -4 & \frac{7}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$$
(%o33) $$\begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i34) cp_kaidan(B2);
```

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -2 & 4 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
(%o34) $$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i40) /* dim W(2) = 2, dim V(2) = 1, a2 in W \ V */
          a2 : matrix([2], [1], [0]);
```

(%o40) $$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

```
(%i41) /* a1 := B . a2 in V */
          a1 : B . a2;
```

(%o41) $$\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

```
(%i42) cp_kaidan(C);
```

$$\begin{pmatrix} 1 & -3 & 2 \\ -3 & 11 & -7 \\ -5 & 19 & -12 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -1 \\ -5 & 19 & -12 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -1 \\ 0 & 4 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 4 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

(%o42) $$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i43) /* c1 in V(3) = W(3) */
          c1 : matrix([-1], [1], [2]);
```

(%o43) $$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

```
(%i44) /* AA := (a1, a2, c1) is a basis of R^3 = W(2) oplus V(3) */
          AA : addcol(a1, a2, c1);
```

(%o44) $$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 1 & 1 \\ 9 & 0 & 2 \end{pmatrix}$$

```
(%i45) /* as a matrix Q := AA */
          Q : AA;
```

(%o45) $$\begin{pmatrix} 3 & 2 & -1 \\ 6 & 1 & 1 \\ 9 & 0 & 2 \end{pmatrix}$$

```
(%i46) P : myinv(AA);
```

$$\begin{pmatrix} 3 & 2 & -1 & 1 & 0 & 0 \\ 6 & 1 & 1 & 0 & 1 & 0 \\ 9 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 6 & 1 & 1 & 0 & 1 & 0 \\ 9 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 9 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & -6 & 5 & -3 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & 5 & -3 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & -\frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & 5 & -3 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & -\frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{3} & -\frac{1}{9} & \frac{2}{9} & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{9} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{9} & -\frac{4}{9} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{5}{3} & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{pmatrix}$$

(%o46)
$$\begin{pmatrix} \frac{2}{9} & -\frac{4}{9} & \frac{1}{3} \\ -\frac{1}{3} & \frac{5}{3} & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

```
(%i47) /* J : The Jordan form of A */
          J : P . A . Q;
```

(%o47)
$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

```
(%i50) S : matrix([2, 0, 0], [0, 2, 0], [0, 0, 3]);
```

(%o50)
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

```
(%i52) N : J - S;
```

(%o52)
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
(%i53) /* AN := A^n = Q(PAQ)^nP = Q(S + N)^nP,
          (S + N)^n = S^n + n*S^(n-1)N + n(n-1)/2*S^(n-2)N^2 + ... + N^n,
          but N^2 = N^3 = ... = N^n = 0. Therefore
          for n >= 2, A^n = Q(S^n + n*S^(n-1)N)P  */
          SN : matrix([2^n, 0, 0], [0, 2^n, 0], [0, 0, 3^n]);
```

(%o53)
$$\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$$

```
(%i56) SN1 : matrix([2^(n-1), 0, 0], [0, 2^(n-1), 0], [0, 0, 3^(n-1)]);
```

(%o56)
$$\begin{pmatrix} 2^{n-1} & 0 & 0 \\ 0 & 2^{n-1} & 0 \\ 0 & 0 & 3^{n-1} \end{pmatrix}$$

```
(%i57) AN : Q . (SN + n * SN1 . N) . P;
```

(%o58)
$$\begin{pmatrix} -\frac{2^{n+1}}{3}+3\left(\frac{2^{n+1}}{9}-\frac{n\,2^{n-1}}{3}\right)+3^n & 3\left(\frac{5\,n\,2^{n-1}}{3}-\frac{2^{n+2}}{9}\right)+\frac{5\,2^{n+1}}{3}-2\,3^n & -2^{n+1}+3^n+3\left(\frac{2^n}{3}-n\,2^{n-1}\right) \\ 6\left(\frac{2^{n+1}}{9}-\frac{n\,2^{n-1}}{3}\right)-3^n-\frac{2^n}{3} & 6\left(\frac{5\,n\,2^{n-1}}{3}-\frac{2^{n+2}}{9}\right)+2\,3^n+\frac{5\,2^n}{3} & -3^n-2^n+6\left(\frac{2^n}{3}-n\,2^{n-1}\right) \\ 9\left(\frac{2^{n+1}}{9}-\frac{n\,2^{n-1}}{3}\right)-2\,3^n & 9\left(\frac{5\,n\,2^{n-1}}{3}-\frac{2^{n+2}}{9}\right)+4\,3^n & 9\left(\frac{2^n}{3}-n\,2^{n-1}\right)-2\,3^n \end{pmatrix}$$

```
(%i59) ratsimp(AN);
```

$$(\%o59)\quad \begin{pmatrix} \frac{2\,3^n - n\,2^n}{2} & -\frac{4\,3^n + (-5\,n-4)\,2^n}{2} & \frac{2\,3^n + (-3\,n-2)\,2^n}{2} \\ (1-n)\,2^n - 3^n & 2\,3^n + (5\,n-1)\,2^n & (1-3\,n)\,2^n - 3^n \\ -\frac{4\,3^n + (3\,n-4)\,2^n}{2} & \frac{8\,3^n + (15\,n-8)\,2^n}{2} & -\frac{4\,3^n + (9\,n-6)\,2^n}{2} \end{pmatrix}$$

```
(%i60)
```